SCIENTIFIC ISSUES

JAN DŁUGOSZ UNIVERSITY
in CZĘSTOCHOWA

MATHEMATICS XVII

2012
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ISBN 1896–0286
ISSN 978–83–7455–285–1

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Publishing House of Jan Długosz University in Częstochowa, ul Waszyngtona 4/8, tel 0048343784329, fax 0048343784319
www.ajd.czest.pl e-mail: wydawnictwo@ajd.czest.pl
Editor’s e-mail: sci.iss.math@ajd.czest.pl
PRACE NAUKOWE

AKADEMIA im. JANA DŁUGOSZA
w CZĘSTOCHOWIE

MATEMATYKA XVII

2012
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UNIFORMLY CONTINUOUS COMPOSITION OPERATOR IN THE SPACE OF FUNCTIONS OF TWO VARIABLES OF BOUNDED $\Phi$-VARIATION IN THE SENSE OF SCHRAMM

WADIE AZIZ, TOMAS EREÜ, NELSON MERENTES, JOSE L. SANCHEZ, MAŁGORZATA WRÓBEL

Abstract

We prove in this paper that if the composition operator $H$, generated by a function $h : I_{b}^{a} \times C \to Y$, maps $\Phi_{1}BV(I_{b}^{a}, C)$ into $\Phi_{2}BV(I_{b}^{a}, Y)$ and is uniformly continuous, then the left-left regularization $h^{*}$ of $h$ is an affine function with respect to the third variable.

1. Introduction

Let $I_{b}^{a}$ denote the rectangle $[a_{1}, b_{1}] \times [a_{2}, b_{2}]$. Let $(X, | \cdot |), (Y, | \cdot |)$ be real normed spaces and $C$ be a convex cone in $X$. For a function $h : I_{b}^{a} \times C \to Y$, denote by $X_{b}^{I_{a}^{b}}$ the algebra of all functions $f : I_{a}^{b} \to X$ and by $H : X_{b}^{I_{a}^{b}} \to Y_{b}^{I_{a}^{b}}$ the Nemytskij operator generated by the function $h$ defined by

$$(Hf)(t, s) = h(t, s, f(t, s)), \quad f \in X_{b}^{I_{a}^{b}}, (t, s) \in I_{a}^{b}.$$ 

Let $(\Phi BV(I_{a}^{b}, X), \| \cdot \|_{\Phi})$ be a Banach space of functions $f \in X_{b}^{I_{a}^{b}}$ which have bounded $\Phi$-variation in the sense of Schramm, where the norm $\| \cdot \|_{\Phi}$ is defined with the aid of Luxemburg-Nakano-Orlicz seminorm [1, 16, 17].

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Assume that $H$ maps the set of functions $f \in \Phi^0BV(I_{a}^{b},X)$ such that $f(I_{a}^{b}) \subset C$ into $\Phi^0BV(I_{a}^{b},Y)$. In the present paper, we prove that, if $H$ is uniformly continuous, then the left-left, right-right, left-right and right-left regularizations of its generator $h$ with respect to first two variables are affine functions with respect to the third variable. This extends the main results of [2] and [14]. In some spaces the representation theorems for the Lipschitzian Nemytskij operators have been established before, see [3-8, 11].

2. Preliminaries

In this section we recall some facts which will be in need in further considerations.

Denote by $\mathbb{R}$ the set of all real numbers and put $\mathbb{R}^+ = [0, \infty)$. We say that a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\varphi$-function if $\varphi$ is continuous on $\mathbb{R}^+$, $\varphi(0) = 0$, $\varphi$ is increasing on $\mathbb{R}^+$ and $\varphi(t) \to \infty$ when $t \to \infty$. Let us recall first the concept of the bounded $\varphi$-variation in the sense of Wiener ([12]). Namely, we say that a function $u : [a, b] \to \mathbb{R}$ has a $\varphi$-bounded variation in the Wiener sense with respect to a $\varphi$-function $\varphi$ provided the quantity $V_{\varphi}^W(u)$ defined by

$$V_{\varphi}^W(u) = V_{\varphi}^W(u; [a, b]) = \sup_{\pi} \sum_{j=1}^{n} \varphi(\left|u(t_j) - u(t_{j-1})\right|)$$

is finite. Here the supremum is taken over all partitions $\pi$ of the interval $[a, b]$.

Next, let $\Phi = \{\phi_n\}$ be a sequence of increasing convex functions, defined on the set of nonnegative real numbers and such that $\Phi_n(0) = 0$ and $\Phi_n(t) > 0$ for $t > 0$ and $n = 1, 2, \ldots$. We say that $\Phi$ is $\Phi^*$-sequence if $\phi_{n+1}(t) \leq \phi_n(t)$ for all $n$, $t$ and $\Phi$-sequences and in addition

$$\sum_{n=1}^{\infty} \phi_n(t) \text{ diverges for all } t > 0. \quad (1)$$

If $\Phi$ is either a $\Phi^*$-sequence or a $\Phi$-sequence, we say that a function $u$ is of $\Phi$-bounded variation in the Schramm sense if the $\Phi$-sum $\sum \phi_n(\left|u(I_n)\right|)$ is finite for any non-overlapping collection $\{I_n\}$ of $I$ ([10]). If $I_n = [a_n, b_n]$ is a subinterval of the interval $I$ ($n = 1, 2, \ldots$) we write $u(I_n) := u(b_n) - u(a_n)$.

We introduce the $\Phi = \{\phi_{n,m}\}$ two dimensional sequence of increasing convex functions, such that $\phi_{n,m}(0) = 0$ and $\phi_{n,m}(t) > 0$ for $t > 0$ and
We say that $\Phi$ is $\Phi$-sequence [13, 14] if
\[
\phi_{n',m'}(t) \leq \phi_{n,m}(t) \quad \text{for each } n' \leq n, \ m' \leq m, \ t \in [0, \infty)
\]
and
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m}(t) \quad \text{diverges for } t > 0.
\]

3. Notation, definitions and auxiliary facts

At the beginning assume that $a = (a_1, a_2)$, $b = (b_1, b_2)$ are two fixed points in the plane $\mathbb{R}^2$. Denote by $I_{a}^b$ the rectangle generated by the points $a$ and $b$, i.e., $I_{a}^b = [a_1, b_1] \times [a_2, b_2]$.

Next, let us assume that $\{I_n\}$ and $\{J_m\}$ are two sequences of closed subintervals of the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively. It means $I_n = [a_n^1, b_n^1]$, $(n = 1, 2, \ldots)$, $J_m = [a_m^2, b_m^2]$, $(m = 1, 2, \ldots)$.

Finally assume that $f : I \to \mathbb{R}$ is a given function and let $\Phi = \{\phi_{n,m}\}$ be a fixed double $\Phi$ sequence.

Fix $x_2 \in J_1 = [a_2, b_2]$ and consider the function $f(\cdot, x_2) : [a_1, b_1] \to \mathbb{R}$. The quantity $V_{\phi_n,m}^{S,I_1}$ defined by the formula
\[
V_{\phi_n,m}^{S,I_1}(u) = \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}(|f(I_n, x_2)|)
= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}(|f(b_n, x_2) - f(a_n, x_2)|)
= \sup_{\pi_1} \sum_{n=1}^{\infty} \phi_{n,m}(|f(b_n, x_2) - f(a_n, x_2)|),
\]
is said to be $\Phi$-variation in the sense of Schramm of the function $f(\cdot, x_2)$. In the case when $V_{\phi_n,m}^{S,I_1}(f) < \infty$ we will say that $f$ has a bounded $\Phi$-variation in the sense of Schramm with respect to the first variable (with fixed the second one). In the same way one can define the concept of the $\Phi$-variation of the function $f(x_1, \cdot)$ in the Schramm sense. It is denoted by $V_{\phi_n,m}^{S,J_1}$. Obviously, if $V_{\phi_n,m}^{S,J_1}(f) < \infty$ then one can say that $f$ has bounded $\Phi$-variation in the sense of Schramm with respect to the second variable (with fixed the first one).

Let us pay attention to the fact that the least upper bound in formula (3) is taken with respect to all sequences $\{I_n\}$ of subintervals of the interval $I_1$.
quantity $V_{\Phi, I_1}^S$ \cite{13,14}. Further, we provide the definition of the concept of two dimensional (or bi-dimensional) variation in the sense of Schramm.

**Definition 1.** The quantity $V_{\Phi, J_1}^S(f)$ defined by the formula

$$V_{\Phi, J_1}^S(f) = \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} \left( |f(I_n, J_m)| \right) =$$

$$= \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} \left( |f(b_n, J_m) - f(a_n, J_m)| \right) =$$

$$= \sup_{\pi_1, \pi_2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} \left( |f(a_n, c_m) + f(b_n, d_m) - f(a_n, d_m) - f(b_n, c_m)| \right),$$

is said to be the bi-dimensional variation in the sense of Schramm of the function $f$ where the least upper bound is considered on all collections of closed and bounded subintervals $\{I_n\}, \{J_m\}$ of intervals $I_1$ and $J_1$ respectively.

Finally, we introduce the definition of the main considered concept.

**Definition 2.** We say that the quantity $TV_{\Phi}^S(f)$ defined by the formula

$$TV_{\Phi}^S(f) = V_{\Phi, I_1}^S(f) + V_{\Phi, J_1}^S(f) + V_{\Phi, I_1}^{S, f}(f)$$

is the total $\Phi$-variation of the function $f$ in the sense of Schramm.

A function $f$ is referred as a function with bounded total $\Phi$-variation provided $TV_{\Phi}^S(f) < \infty$.

By $\Phi BV(I_a^b)$ we denote the set of all functions $f : I_a^b \to X$ which have bounded total $\Phi$-variation in the sense of Schramm.

By $P_{\Phi}$ let us denote the functional defined on the set $\Phi BV(I_a^b)$ in the following way:

$$P_{\Phi}(f) = \inf \left\{ \epsilon > 0 : TV_{\Phi}^S \left( \frac{f}{\epsilon} \right) \leq 1 \right\}. \quad (4)$$

The main result in \cite{13} asserts that the set $\Phi BV(I_a^b)$ forms a Banach algebra with the norm defined by the formula

$$\|f\|_{\Phi} = |f(a)| + P_{\Phi}(f). \quad (5)$$

**Observation 1.** If we take the $\Phi$-sequence defined as follows

$$\Phi = \{ \phi_{n,m} : \phi_{n,m}(t) = t^p; \ 1 < p < \infty, n, m = 1, 2, ... \}$$

then we can check that $P_{\Phi}(f) = (TV_{\Phi}^S(f))^{1/p}$. 
Our next result is contained in the following Lemma.

**Lemma 1.** Let \( f \in \Phi BV \left( I^b_a, X \right) \) and \( \Phi \in \Phi^* \). Then \( f \) has the following properties:

1. If \((t, s), (t', s') \in I^b_a\) then \( |f(t, s) - f(t', s')| \leq 4\Phi^{-1}_{n,m} \left( \frac{1}{2} \right) P_\Phi(f) \).
2. If \( P_\Phi(f) > 0 \) then \( TV^\Phi(f/P_\Phi(f)) \leq 1 \).
3. Let \( r > 0 \). Then \( TV^\Phi(f/r) \leq 1 \) if and only if \( P_\Phi(f) \leq r \).

**Observation 2.** From part (1) of Lemma 1, we deduced that each function \( f \in \Phi BV \left( I^b_a, X \right) \) is bounded. Moreover, the following estimation is satisfied

\[
\|f\|_\infty = \sup \left\{ |f(t, s)| : (t, s) \in I^b_a \right\} \leq |f(a)| + 4\Phi^{-1}_{n,m} \left( \frac{1}{2} \right) P_\Phi(f)
\]

if \( n, m = 1, 2, \ldots \) where the symbol \( \|f\|_\infty \) denotes the supremum norm, i.e.

\[
\|f\|_\infty = \sup \left\{ |f(t, s)| : (t, s) \in I^b_a \right\}
\]

Let us fix arbitrary \( f \in \Phi BV \left( I^b_a, X \right) \). Then the function \( f^* : I^b_a \to X \) defined by formula

\[
f^* (x_1, x_2) = \left\{ \begin{array}{ll}
\lim_{(y_1, y_2) \to (x_1-0, x_2-0)} f(y_1, y_2), & (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2],
\lim_{(y_1, y_2) \to (x_1-0, x_2+0)} f(y_1, y_2), & x_1 \in (a_1, b_1] \text{ and } x_2 = a_2,
\lim_{(y_1, y_2) \to (a_1+0, x_2-0)} f(y_1, y_2), & x_1 = a_1 \text{ and } x_2 \in (a_2, b_2],
\lim_{(y_1, y_2) \to (a_1+0, a_2+0)} f(y_1, y_2), & x_1 = a_1 \text{ and } x_2 = a_2
\end{array} \right.
\]

is called the left-left regularization of the function \( f \). The existence of all one-sided limits used above was proved in [15].

**Definition 3.** A function \( f : I^b_a \to \mathbb{R} \) is said to be left-left continuous if

\[
\lim_{y_1 \to x_1-0, y_2 \to x_2-0} f(y_1, y_2) = f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in (a_1, b_1] \times (a_2, b_2].
\]

By \( \Phi BV^* \left( I^b_a \right) \) is denoted the subspace of \( \Phi BV \left( I^b_a \right) \) consisting of those functions which are left-left continuous on \((a_1, b_1] \times (a_2, b_2]\) and by \( \mathcal{L}(X, Y) \) the space defined by

\[
\mathcal{L}(X, Y) := \{ f : X \to Y : f \text{ is linear} \}
\]

**Lemma 2 ([14]).** If \( f \in \Phi BV \left( I^b_a \right) \), then \( f^* \in \Phi BV^* \left( I^b_a \right) \).

In the sequel we are going to deal with the main result of this paper.
Our main result reads as follows:

**Theorem 1.** Let \( I^b_a \subset \mathbb{R}^2 \) be a rectangle, \((X, | \cdot |_X)\) be a real normed space, \((Y, | \cdot |_Y)\) be a real Banach space, \(C\) be a convex cone in \(X\). If the composition operator \( H \) generated by \( h : I^b_a \times C \rightarrow Y \) transforms \( \Phi_1 BV \left( I^b_a, C \right) \) into \( \Phi_2 BV \left( I^b_a, Y \right) \) and is uniformly continuous, then there exist functions \( A \in \mathcal{L}(X, Y) \) and \( B \in \Phi_2 BV \left( I^b_a, Y \right) \) such that

\[
h^*(t, s, y) = A(t, s)y + B(t, s), \quad (t, s) \in I^b_a, \quad y \in C,
\]

where \( h^* \) is the left-left regularization of \( h \).

**Proof.** For every \( y \in C \) the constant function \( f(t, s) = y \) with \((t, s) \in I^b_a\) belongs to \( \Phi_1 BV \left( I^b_a, C \right) \). Since \( H \) maps \( \Phi_1 BV \left( I^b_a, C \right) \) into \( \Phi_2 BV \left( I^b_a, Y \right) \), it follows that the function \((t, s) \mapsto h(t, s, y), (t, s) \in I^b_a\), belongs to \( \Phi_2 BV \left( I^b_a, Y \right) \).

Now the completeness of \( \Phi_2 BV \left( I^b_a, Y \right) \) implies the existence of the left-left regularization \( h^* \) of \( h \).

By assumption \( H \) is uniformly continuous on \( \Phi_1 BV \left( I^b_a, C \right) \). Let \( \omega \) be the modulus of continuity of \( H \) that is

\[
\omega(\rho) := \sup \left\{ \| H(f_1) - H(f_2) \|_{\Phi_2} : \| f_1 - f_2 \|_{\Phi_1} \leq \rho; \ f_1, f_2 \in \Phi_1 BV \left( I^b_a, C \right) \right\}
\]

for \( \rho > 0 \). Hence we get

\[
\| H(f_1) - H(f_2) \|_{\Phi_2} \leq \omega(\| f_1 - f_2 \|_{\Phi_1}), \quad \text{for} \quad f_1, f_2 \in \Phi_1 BV \left( I^b_a, C \right). \quad (7)
\]

From the definition of the norm \( \| \cdot \|_{\Phi_2} \) we obtain

\[
P_{\Phi_2} (H(f_1) - H(f_2)) \leq \| H(f_1) - H(f_2) \|_{\Phi_2}, \quad \text{for} \quad f_1, f_2 \in \Phi_1 BV \left( I^b_a, C \right). \quad (8)
\]

In view of (8), Definitions 1., 2. and Lemma 1.(3), if \( \omega(\| f_1 - f_2 \|_{\Phi}) > 0 \), then

\[
V^S_{\Phi, I^b_a}(f) \left( \frac{(H(f_1) - H(f_2))(\cdot, a_2)}{\omega(\| f_1 - f_2 \|_{\Phi_1})} \right) \leq T V^S_{\Phi^2} \left( \frac{H f_1 - H f_2}{\omega(\| f_1 - f_2 \|_{\Phi_1})} \right) \leq 1. \quad (9)
\]

The definitions of the operator \( H \) and the functional \( V^S_{\Phi, I^b_a}(f) \) imply that for any

\[
a_1 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n \leq b_1,
\]

and

\[
a_2 \leq \bar{\alpha}_1 < \bar{\beta}_1 < \bar{\alpha}_2 < \bar{\beta}_2 < \cdots < \bar{\alpha}_m < \bar{\beta}_m \leq b_2,
\]
if \( n, m \in \mathbb{N} \), the inequality
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j} \left( |h(\alpha, \pi, f_{1}(\alpha, \pi, x)) - h(\alpha, \pi, f_{2}(\alpha, \pi, x)) + h(\alpha, \pi, f_{1}(\alpha, \pi, y))| \right) \\
\leq 1.
\]
holds.

For \( \alpha, \beta \in \mathbb{R} \), \( \alpha < \beta \), we define functions \( \eta_{\alpha, \beta} : \mathbb{R} \to [0, 1] \) by the following formula:
\[
\eta_{\alpha, \beta}(t) := \begin{cases} 
0 & \text{if } t \leq \alpha \\
\frac{t - \alpha}{\beta - \alpha} & \text{if } \alpha \leq t \leq \beta \\
1 & \text{if } \beta \leq t. 
\end{cases}
\]

First let us fix \( t \in (a_{1}, b_{1}], s \in (a_{2}, b_{2}] \) and \( n, m \in \mathbb{N} \). For arbitrary sequences
\[
a_{1} \leq \alpha_{1} < \beta_{1} < \alpha_{2} < \beta_{2} < \ldots < \alpha_{n} < \beta_{n} \leq t
\]
\[
a_{2} \leq \alpha_{1} < \beta_{1} < \alpha_{2} < \beta_{2} < \ldots < \alpha_{m} < \beta_{m} \leq s
\]
and \( y_{1}, y_{2} \in C, y_{1} \neq y_{2} \) the functions \( f_{1}, f_{2} : I \to X \) defined by
\[
f_{e}(\tau, \gamma) := \frac{1}{2} \left[ (\eta_{\alpha_{1}, \beta_{1}}(\tau) + \eta_{\alpha_{2}, \beta_{2}}(\gamma) - 1)(y_{1} - y_{2}) + y_{1} + y_{2} \right],
\]
for every \( (\tau, \gamma) \in I_{a_{e}}, \ell = 1, 2 \) belong to the space \( \Phi_{1}BV(I_{a_{e}}, C) \). From this we infer that
\[
f_{1}(\cdot, \cdot) - f_{2}(\cdot, \cdot) = \frac{y_{1} - y_{2}}{2},
\]
therefore
\[
\|f_{1} - f_{2}\|_{\Phi} = \frac{|y_{1} - y_{2}|}{2};
\]
moreover
\[
f_{1}(\alpha_{i}, \overline{\alpha}_{i}) = y_{2}; \quad f_{2}(\alpha_{i}, \overline{\alpha}_{i}) = -\frac{y_{1} + 3y_{2}}{2};
\]
\[
f_{1}(\alpha_{i}, \overline{\beta}_{j}) = \frac{y_{1} + y_{2}}{2}; \quad f_{2}(\alpha_{i}, \overline{\beta}_{j}) = y_{2},
\]
\[
f_{1}(\beta_{i}, \overline{\alpha}_{i}) = y_{2}; \quad f_{2}(\beta_{i}, \overline{\alpha}_{i}) = -\frac{y_{1} + 3y_{2}}{2};
\]
\[
f_{1}(\beta_{i}, \overline{\beta}_{j}) = x_{1}; \quad f_{2}(\beta_{i}, \overline{\beta}_{j}) = \frac{y_{1} + y_{2}}{2}.
\]
Applying (10), we get
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j} \left( \frac{h(\alpha_i, \tau_j, y_2) - h(\alpha_i, \tau_j, y_1 + y_2) - h(\alpha_i, \tau_j, y_2) + h(\alpha_i, \tau_j, y_1)}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right) \\
+ \frac{-h(\beta_i, \tau_j, y_2) + h(\beta_i, \tau_j, y_1 + y_2) + h(\beta_i, \tau_j, y_2) + h(\beta_i, \tau_j, y_1)}{\omega(\|f_1 - f_2\|_{\Phi_1})} \right) \leq 1. \tag{13}
\]

In view of continuity of \(\phi_{i,j}\), and Lemma 2., the left-left continuity of \(h^*\) we infer that
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}(x) \leq 1 \quad \text{for} \ n, m = 1, 2, \ldots, \tag{14}
\]
where
\[
x = \frac{h^*(t, s, y_1) - 2h^*(t, s, y_1 + y_2) + h^*(t, s, y_2)}{\omega \left( \left| \frac{y_1 - y_2}{2} \right| \right)}.
\]
Since \(n, m \in \mathbb{N}\) are arbitrary, condition (14) implies inequality
\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \phi_{i,j}(x) \leq 1.
\]
In view of (2) we get \(x = 0\), i.e.
\[
h^* \left( t, s, \frac{y_1 + y_2}{2} \right) = \frac{h^*(t, s, y_1) + h^*(t, s, y_2)}{2} \tag{15}
\]
for all \((t, s) \in (a_1, b_1) \times (a_2, b_2)\) and \(y_1, y_2 \in C\).

For \(t \in (a_1, b_1)\) and \(s = b_2\) let
\[
a_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n < t
\]
and
\[
a_2 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m < b_2.
\]
Proceeding as above we get (13).

If \(\alpha_1 \uparrow t\) and \(\beta_m \downarrow s\) in (13), then we get (15).

The cases when \(t = a_1\) and \(s \in (a_2, b_2)\) or \(t = a_1\) and \(s = a_2\) can be treated similarly. Consequently
\[
h^* \left( t, s, \frac{y_1 + y_2}{2} \right) = \frac{h^*(t, s, y_1) + h^*(t, s, y_2)}{2}
\]
is valid for all \((t, s) \in I_n^b\) and all \(y_1, y_2 \in C\).
Therefore, the function \( h^*(t, s, \cdot) \) satisfies the Jensen functional equation in \( C \) for \((t, s) \in I^b_a\). Modifying the standard argument (Kuczma [9]), we conclude that for each \((t, s) \in I^b_a\) there exist additive functions \( A(t, s) : C \to \mathcal{L}(X, Y) \) and \( B(t, s) \in Y \) such that

\[
h^*(\cdot, y) = A(\cdot)y + B(\cdot), \quad y \in C.
\]  

(16)

The uniform continuity of the operator \( H : \Phi_1 BV (I^b_a, C) \to \Phi_2 BV (I^b_a, Y) \) implies the continuity of the additive function \( A(t, s) \).

Consequently \( A(t, s) \in \mathcal{L}(X, Y) \).

Finally, notice that \( A(t, s)(0) = \{0\} \) for every \((t, s) \in I^b_a\). Therefore, putting \( y = 0 \) in (16), we get

\[
h^*(t, s, 0) = B(t, s), \quad (t, s) \in I^b_a,
\]

which implies \( B \in \Phi_2 BV (I^b_a, Y) \).

\[\square\]

Observation 3. A similar theorem to Theorem 1. is valid for the right-right, right-left and left-right regularizations of \( h(\cdot, y), y \in C \).

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ON SOME ADDITION FORMULAS FOR HOMOGRAPHIC TYPE FUNCTIONS

KATARZYNA DOMAŃSKA

ABSTRACT
We deal with the functional equation (so called addition formula) of the form

\[ f(x + y) = F(f(x), f(y)), \]

where \( F \) is an associative rational function. The class of associative rational functions was described by A. Chéritat [1] and his work was followed by a paper of the author. For function \( F \) defined by

\[ F(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)), \]

where \( \varphi \) is a homographic function, the addition formula is fulfilled by homographic type functions.

We consider the class of the associative rational functions defined by formula

\[ F(u, v) = \frac{uv}{\alpha uv + u + v}, \]

where \( \alpha \) is a fixed real number.

1. INTRODUCTION

For the rational two-place real-valued function \( F \) given by

\[ F(x, y) = \varphi^{-1}(\varphi(x) + \varphi(y)), \hfill (H) \]

where \( \varphi \) is a homographic function, the addition formula has the form

\[ \varphi(f(x + y)) = \varphi(f(x)) + \varphi(f(y)) \]

and it is a conditional functional equation if the domain of \( \varphi \) is not equal to \( \mathbb{R}^2 \). Solutions of the above conditional equation are homographic type functions.
If $F$ is associative and rational then the form (H) states that it belongs to one of the following classes (all of them are considered in their natural domains):

\[ F(u, v) = \frac{uv}{\alpha uv + u + v} \]

\[ F(u, v) = \frac{u + v + 2\lambda uv}{1 - \lambda^2 uv} \]

\[ F(u, v) = \frac{uv - \lambda^2}{u + v + 2\lambda} \]

\[ F(u, v) = \frac{(1 - 2\lambda)uv - \frac{\lambda}{\mu}(u + v) - \frac{1}{\mu^2}}{\lambda uv + u + v + \frac{2 - \lambda}{\mu}} \]

where $\alpha \in \mathbb{R}$, $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ (it is a consequence of the associativity).

Let $\alpha \in \mathbb{R}$ be arbitrary fixed. We consider the rational function

\[ F : \{(x, y) \in \mathbb{R}^2 : \alpha xy + x + y \neq 0\} \longrightarrow \mathbb{R} \]

of the form

\[ F(u, v) = \frac{uv}{\alpha uv + u + v}. \]

It is a rational two-place real-valued function defined on a disconnected subset of the real plane $\mathbb{R}^2$, which satisfies the equation

\[ F(F(x, y), z) = F(x, F(y, z)) \]

for all $(x, y, z) \in \mathbb{R}^3$ such that

\[ \alpha xy + x + y, \alpha yz + y + z, \alpha F(x, y)z + F(x, y) + z, \alpha xF(y, z) + x + F(y, z) \]

are not equal to 0. We shall determine all functions $f : G \longrightarrow \mathbb{R}$, where $(G, \star)$ is a group, which satisfy the functional equation

\[ f(x \star y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)}. \quad (1) \]

A neutral element of a group $(G, \star)$ will be written as 0.

By a solution of the functional equation (1) we understand any function $f : G \longrightarrow \mathbb{R}$ which satisfies equality (1) for every pair $(x, y) \in G^2$ such that $\alpha f(x)f(y) + f(x) + f(y) \neq 0$. Thus we deal with the following conditional functional equation:

\[ \alpha f(x)f(y) + f(x) + f(y) \neq 0 \]
implies
\[ f(x \ast y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)} \] (E)
for all \( x, y \in G \).

Some results on addition formulas can be found for example in the work of K. Domanińska and R. Ger [2].

The following lemma will be useful in the sequel (see R. Ger [4]).

**Lemma.** (On characterization of subgroups). Let \((G, \ast)\) be a group. Then \((H, \ast)\) is a subgroup of the group \((G, \ast)\) if and only if \(G \supset H \neq \emptyset\) and
\[ H \ast H' \subset H', \]
where \(H' := G \setminus H\).

**2. Main results**

We proceed with a description of solutions of (E) if \(\alpha = 1\).

**Theorem 1.** Let \((G, \ast)\) be a group. A function \(f : G \rightarrow \mathbb{R}\) yields a nonconstant solution to the functional equation
\[ f(x)f(y) + f(x) + f(y) \neq 0 \]
implies
\[ f(x \ast y) = \frac{f(x)f(y)}{f(x)f(y) + f(x) + f(y)} \] (E1)
for all \(x, y \in G\), if and only if either
\[ f(x) := \begin{cases} -2 & \text{if } x \in H, \\ 0 & \text{if } x \in G \setminus H \end{cases} \]
or
\[ f(x) := \begin{cases} -1 & \text{if } x \in \Gamma, \\ -2 & \text{if } x \in G \setminus \Gamma \end{cases} \]
or
\[ f(x) = \frac{1}{A(x) - 1}, \quad x \in G \]
where \((H, \ast), (\Gamma, \ast)\) are subgroups of the group \((G, \ast)\), and \(A : G \rightarrow \mathbb{R}\) is a homomorphism such that \(1 \notin A(G)\).
Proof. Assume that $f$ is a nonconstant solution of the equation (E1). First we show that $f(0) \in \{-2, -1, 0\}$. Indeed, setting $x = y = 0$ in (E1) we obtain

$$f(0)^2 + 2f(0) = 0 \quad \text{or} \quad f(0) = \frac{f(0)^2}{f(0)^2 + 2f(0)},$$

hence $f(0) \in \{-2, -1, 0\}$.

First assume that $f(0) = -2$. We show that $f(G) \subset \{-2, 0\}$. In fact, putting $y = 0$ in (E1) we obtain

$$-2f(x) + f(x) - 2 = 0 \quad \text{or} \quad f(x) = \frac{-2f(x)}{-2f(x) + f(x) - 2}$$

for all $x \in G$. Consequently

$$f(x) = -2 \quad \text{or} \quad f(x) = \frac{2f(x)}{f(x) + 2}$$

for all $x \in G$ and since the equality

$$c = \frac{2c}{c + 2}$$

forces $c$ to vanish, we infer that

$$f(x) = -2 \quad \text{or} \quad f(x) = 0$$

for all $x \in G$. Since $f$ is assumed to be nonconstant, both the complementary sets

$$H := \{x \in G : f(x) = -2\} \quad \text{and} \quad H' = \{x \in G : f(x) = 0\}$$

are nonempty.

We shall show that $H \ast H' \subset H'$, which implies that $H$ is a subgroup of the group $G$ (see Lemma). Fix arbitrarily elements $x \in H$ and $y \in H'$. Since $f(x)f(y) + f(x) + f(y) = -2$, we get $f(x \ast y) = 0$ by (E1) i.e. $x \ast y \in H'$, which was to be shown. So, in this case we have

$$f(x) := \begin{cases} \ -2 & \text{if } x \in H, \\ \ 0 & \text{if } x \in G \setminus H. \end{cases}$$

Let now $f(0) = -1$. Assume that $f(a) = 0$ for some $a \neq 0$. Putting $x = 0$ and $y = a$ in (E1) we get $f(0) = 0$ which leads to a contradiction. Consequently, in this case we have $f(x) \neq 0$ for all $x \in G$. We define a function $A : G \longrightarrow \mathbb{R}$ by the formula

$$A(x) = \frac{1}{f(x)} + 1, \quad x \in G.$$
Observe that $1 \not\in A(G)$. A straightforward verification shows that

$$f(x)f(y) + f(x) + f(y) = 0$$

if and only if $A(x) + A(y) = 1$

for all $x, y \in G$. Thus jointly with (E1) we infer that

$$A(x) + A(y) \neq 1$$

implies

$$A(x \star y) = 1 + \frac{f(x)f(y) + f(x) + f(y)}{f(x)f(y)} = 2 + \frac{1}{f(x)} + \frac{1}{f(y)} = A(x) + A(y)$$

which states that the function $A$ yields a solution of the equation

$$g(x) + g(y) \neq 1 \text{ implies } g(x \star y) = g(x) + g(y)$$

for all $x, y \in G$.

Since $f(0) = -1$, evidently $A(0) = 0$. From the theorem proved by R. Ger [3] (since $A(0) = 0$) we conclude that $A$ yields a homomorphism of groups $G$ and $\mathbb{R}$ or there exists a subgroup $\Gamma$ of the group $G$ such that $A$ is of the form

$$A(x) := \begin{cases} 0 & \text{if } x \in \Gamma, \\ \frac{1}{2} & \text{if } x \in G \setminus \Gamma. \end{cases}$$

Accordingly,

$$f(x) = \frac{1}{A(x) - 1}, \quad x \in G$$

or

$$f(x) := \begin{cases} -2 & \text{if } x \in \Gamma, \\ 0 & \text{if } x \in G \setminus \Gamma. \end{cases}$$

Now let $f(0) = 0$. Putting $y = 0$ in (E1) we obtain that $f(x) = 0$ for all $x \in G$, contradicting the assumption that $f$ is nonconstant. It is easy to check that each of those functions yields a solution to the equation (E1). Thus the proof has been completed. \(\square\)

Now we proceed with a description of solutions of (E).

**Theorem 2.** Let $(G, \star)$ be a group and $\alpha \in \mathbb{R} \setminus \{0\}$ be fixed. A function $f : G \rightarrow \mathbb{R}$ yields a nonconstant solution to the functional equation

$$\alpha f(x)f(y) + f(x) + f(y) \neq 0$$
implies
\[ f(x \star y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)} \quad (E) \]

for all \( x, y \in G \) if and only if either
\[
f(x) := \begin{cases} 
-\frac{2}{\alpha} & \text{if } x \in H, \\
0 & \text{if } x \in G \setminus H
\end{cases}
\]
or
\[
f(x) := \begin{cases} 
-\frac{1}{\alpha} & \text{if } x \in \Gamma \\
-\frac{2}{\alpha} & \text{if } x \in G \setminus \Gamma
\end{cases}
\]
or
\[ f(x) = \frac{1}{\alpha(A(x) - 1)}, \quad x \in G \]
where \((H, \star), (\Gamma, \star)\) are subgroups of the group \((G, \star)\), and \(A : G \to \mathbb{R}\) is a homomorphism such that \(1 \notin A(G)\).

Proof. Let \(\alpha\) be arbitrarily fixed nonzero number. Assume that \(f\) is a non-constant solution of the equation \((E)\) i.e.
\[ \alpha f(x)f(y) + f(x) + f(y) \neq 0 \implies f(x \star y) = \frac{f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)} \]
for all \(x, y \in G\). Hence
\[ \alpha^2 f(x)f(y) + \alpha f(x) + \alpha f(y) \neq 0 \implies \alpha f(x \star y) = \frac{\alpha f(x)f(y)}{\alpha f(x)f(y) + f(x) + f(y)} \]
for all \(x, y \in G\) i.e.
\[ \alpha f(x)\alpha f(y) + \alpha f(x) + \alpha f(y) \neq 0 \implies \alpha f(x \star y) = \frac{\alpha^2 f(x)f(y)}{\alpha f(x)f(y) + \alpha f(x) + \alpha f(y)} \]
for all \(x, y \in G\). Thus it is easy to observe that \((E)\) states that the function \(g := \alpha f\) satisfies the following functional equation:
\[ g(x)g(y) + g(x) + g(y) \neq 0 \implies g(x \star y) = \frac{g(x)g(y)}{g(x)g(y) + g(x) + g(y)} \]
for all \(x, y \in G\). From the Theorem 1 we conclude that \(g\) is of the form
\[
g(x) := \begin{cases} 
-2 & \text{if } x \in H, \\
0 & \text{if } x \in G \setminus H
\end{cases}
\]
or
\[ g(x) := \begin{cases} -1 & \text{if } x \in \Gamma \\ -2 & \text{if } x \in G \setminus \Gamma \end{cases} \]
or
\[ g(x) = \frac{1}{A(x) - 1}, \quad x \in G \]
where \((H, \cdot), (\Gamma, \cdot)\) are subgroups of the group \((G, \cdot)\), and \(A : \Gamma \to \mathbb{R}\) is a homomorphism such that \(1 \not\in A(\Gamma)\). This states that \(f\) is of the above form.

It is easy to check that each of those functions yields a solution to the equation \((E)\). Thus the proof has been completed. \(\square\)

The following remark gives the forms of constant solutions of the equation \((E)\).

**Remark.** Let \((G, \cdot)\) be a groupoid and \(\alpha\) be a fixed nonzero number. The only constant solutions of \((E)\) are as follows: \(f = -\frac{2}{\alpha}, f = 0\) and \(f = -\frac{1}{\alpha}\).

To check this, assume that \(f = c\) fulfills \((E)\). Then
\[ \alpha c^2 + 2c \neq 0 \quad \text{implies} \quad c = \frac{c^2}{\alpha c^2 + 2c} \]
i.e.
\[ c = 0 \quad \text{or} \quad \alpha c + 2 = 0 \quad \text{or} \quad 1 = \frac{1}{\alpha c + 2}. \]
Hence
\[ c \in \left\{ -\frac{2}{\alpha}, -\frac{1}{\alpha}, 0 \right\}, \]
which was to be shown. \(\square\)

The following theorem gives the form of solutions of the equation \((E)\) if \(\alpha = 0\), i.e. the form of solutions of the following equation
\[ f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x \cdot y) = \frac{f(x)f(y)}{f(x) + f(y)} \quad (E0) \]

**Theorem 3.** Let \((G, \cdot)\) be a monoid. The only solution \(f : G \to \mathbb{R}\) of the equation \((E0)\) is \(f = 0\).

**Proof.** Assume that \(f\) is a solution of the equation \((E0)\). First we show that \(f(0) = 0\). In fact, setting \(x = y = 0\) in \((E1)\) we obtain
\[ f(0) = 0 \quad \text{or} \quad f(0) = \frac{f(0)^2}{2f(0)} = \frac{1}{2} f(0) \]
whence $f(0) = 0$. Fix arbitrarily an $x \in G$ and take $y = 0$. Then, by (E0), we get

$$f(x) \neq 0 \implies f(x) = 0$$

which implies $f = 0$ and which was to be shown. It is easy to check that $f = 0$ yields a solution to the equation (E0). Thus the proof has been completed. □

References


ON UPPER AND LOWER STRONG QUASI-UNIFORM CONVERGENCE OF MULTIVALUED MAPS

IRENA DOMNIK, AGATA SOCHACZEWSKA

Abstract

The concept of strong convergence of functions and multifunctions was introduced by I. Kupka, V. Toma and A. Sochaczewska. In this paper we consider new definitions of convergence for the nets of multifunctions – upper and lower strong quasi-uniform convergence.

1. Introduction

In literature we can find different kinds of convergence of functions and multifunctions in topological spaces. We will introduce these types which are formulated in the terms of open covers and stars.

Definition 1. Let \((Y, \tau)\) be a topological space. The set \(\bigcup\{A \in \mathcal{A} : A \cap E \neq \emptyset\}\) is called a star of a set \(E\) with respect to the cover \(\mathcal{A}\) of the space \(Y\). This set is denoted by \(\text{St}(E, \mathcal{A})\).

The concept of strong convergence of functions was considered by I. Kupka and V. Toma in ([4]).

Definition 2. ([4]) Let \(X\) be an arbitrary set and \((Y, \tau)\) be a topological space. If \(\{f_j : j \in J\}\) is a net of functions from \(X\) to \(Y\), then we say that \(\{f_j : j \in J\}\)
strongly converges to a function \( f : X \to Y \) if for every open cover \( A \) of \( Y \) there exists an index \( j_0 \in J \) such that \( f_j(x) \in \text{St}(f(x), A) \) for every \( x \in X \) and for every \( j \) such that \( j \geq j_0 \).

The next definition of the strong convergence has been introduced by A. Sochaczewska in [6].

**Definition 3.** ([6]) Let \((X, \tau_X), (Y, \tau_Y)\) be topological spaces. Let moreover \( f_n, f : X \to Y \) if \( n \in \mathbb{N} \). We say that a sequence \( (f_n)_{n \in \mathbb{N}} \) is strongly quasi-uniformly convergent to the function \( f \) if it is pointwise convergent and for each open cover \( A \) of \( Y \) and for each \( x_0 \in X \) there exists \( n_0 \in \mathbb{N} \) such that for every \( n, n \geq n_0 \) there exists a neighbourhood \( U \) of \( x_0 \) such that

\[
f_n(x) \in \text{St}(f(x), A)
\]

for all \( x \in U \).

The convergence in the sense of I. Kupka and V. Toma implies the strong quasi-uniform convergence, but not conversely ([6]).

Let \( X \) and \( Y \) be two sets and \( S(Y) \) denote the set of all nonempty subsets of \( Y \). A multivalued function \( F \) from \( X \) into \( Y \) can be considered as a function from \( X \) to \( S(Y) \). Nevertheless, we will write \( F : X \to Y \).

The strong convergence was considered for nets of multivalued maps by I. Kupka in ([3]). Namely:

**Definition 4.** ([3]) Let \((X, \tau_X), (Y, \tau_Y)\) be topological spaces. Let \( \{F_j : j \in J\} \) be a net of multifunctions from \( X \) into \( Y \). Let \( F \) be a multifunction from \( X \) to \( Y \). We say that \( \{F_j : j \in J\} \) strongly converges to \( F \) if for every open cover \( A \) of \( Y \) there exists an index \( j_0 \in J \) such that the inclusions

\[
F(x) \subset \text{St}(F_j(x), A), \quad F_j(x) \subset \text{St}(F(x), A)
\]

hold for all \( j, j \geq j_0 \) and \( x \in X \).

I. Domnik in ([1]) introduced the following kind of convergence of nets of multivalued maps:

**Definition 5.** ([1]) Let \( X \) be a nonempty set and let \((Y, \tau)\) be a topological space. A net \( \{F_j : j \in J\} \) of multivalued maps \( F_j : X \to Y \) is said to be upper (respectively: lower) strongly convergent to a multivalued map \( F : X \to Y \) if
for each open cover $\mathcal{A}$ of $Y$ there exists $j_0 \in J$ such that $F_j(x) \subset \text{St}(F(x), \mathcal{A})$ (respectively: $F(x) \subset \text{St}(F_j(x), \mathcal{A})$) for every $j \in J$, $j \geq j_0$, and $x \in X$.

I. Domnik considered properties of this kind of convergence, for example preservation of continuity ([1]).

2. UPPER AND LOWER STRONG QUASI-UNIFORM CONVERGENCE

In this paper we will consider new types of convergence for the net of multivalued maps — upper and lower strong quasi-uniform convergence.

**Definition 6.** Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces. A net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \to Y$ is said to be upper (respectively: lower) strongly quasi-uniformly convergent to a map $F : X \to Y$ if for every open cover $\mathcal{A}$ of the space $Y$ and for each point $x_0 \in X$ there exists $j_0 \in J$ such that for every $j$, $j \geq j_0$, there exists a neighbourhood $U$ of the point $x_0$ with the inclusion $F_j(x) \subset \text{St}(F(x), \mathcal{A})$ (respectively: $F(x) \subset \text{St}(F_j(x), \mathcal{A})$) for every point $x$ of the set $U$.

Let observe that this kind of convergence is closely connected with the convergence of nets of multivalued maps in the sense of I. Domnik. Moreover, the upper strong quasi-uniform convergence follows from the convergence in the sense of A. Sochaczewska for sequences of functions.

In further part of the article we will apply the following definition and lemmas.

**Definition 7.** ([2]) Let $(Y, \tau_Y)$ be a topological space. A subset $A$ of $Y$ is called $\alpha$-paracompact if for every $\tau_Y$-open cover $\mathcal{A}$ of $A$ there is a $\tau_Y$-open locally finite cover $\mathcal{B}$ such that $\mathcal{B}$ is a refinement of $\mathcal{A}$.

**Remark 1.** ([2]) Every compact set is $\alpha$-paracompact.

**Lemma 1.** ([2]) Every $\alpha$-paracompact subset of a Hausdorff space is closed.

**Lemma 2.** ([1]) Let $(Y, \tau_Y)$ be a regular space. If $A$ is $\alpha$-paracompact subset of $Y$, $U$ is open in $Y$ and $A$ is the subset of $U$, then there exists an open set $V$ satisfying inclusions $A \subset V \subset \text{cl}(V) \subset U$.

For an open subset $U$ of $Y$ we define

$U^+ := \{ B \in S(Y) : B \subset U \}$,

$U^- := \{ B \in S(Y) : B \cap U \neq \emptyset \}$. 
The families

\[ B := \{ U^+ : U \in \tau_Y \}, \quad \mathcal{P} := \{ U^- : U \in \tau_Y \} \]

form a base and subbase of the upper and lower Vietoris topology, respectively. These topologies will be denoted by \( \tau_Y^+ \) and \( \tau_Y^- \).

Let \( F, F_j : X \to Y, j \in J \) be multivalued maps. We will write

\[ F \in (\tau_Y^+) \lim F_j, \quad \text{respectively} \quad F \in (\tau_Y^-) \lim F_j \]

if the net \( \{ F_j : j \in J \} \) is pointwise convergent to \( F \) with respect to topology \( \tau_Y^+ \), respectively to \( \tau_Y^- \).

**Theorem 1.** Let \((X, \tau_X)\) be a topological space, \((Y, \tau_Y)\) — a regular space. If a multivalued map \( F : X \to Y \) has \( \alpha \)-paracompact values, then the upper strong quasi-uniform convergence of a net \( \{ F_j : j \in J \} \) to \( F \) implies \( \tau_Y^+ \)-pointwise convergence.

**Proof.** Let \( x_0 \in X \) and \( V^+ \in B \) be such that \( F(x_0) \in V^+ \). Then \( V \) is an open subset of \( Y \) and \( F(x_0) \subset V \). By the assumption the values of the multivalued map \( F \) are \( \alpha \)-paracompact and \( Y \) is a regular space. It implies the existence of an open subset \( W \) of \( Y \) with properties \( F(x_0) \subset W \subset \text{cl}(W) \subset V \). The family \( A = \{ V, Y \setminus \text{cl}(W) \} \) forms an open cover of the space \( Y \).

The net \( \{ F_j : j \in J \} \) is upper strongly quasi-uniformly convergent to \( F \). It means that for a cover \( A \) there exists \( j_0 \in J \) such that for \( j, j \geq j_0 \) there is a neighbourhood \( U \) of \( x_0 \) such that \( F_j(x) \subset \text{St}(F(x), A) \) holds for every point \( x \) from the set \( U \). Assume \( j \geq j_0 \). In particular, \( x_0 \in U \) and we have the condition \( F_j(x_0) \subset \text{St}(F(x_0), A) = V \). It follows that \( F_j(x_0) \in V^+ \) for all \( j, j \geq j_0 \). Hence we obtain

\[ F \in (\tau_Y^+) \lim F_j. \]

**Theorem 2.** Let \((X, \tau_X), (Y, \tau_Y)\) be topological spaces. If \( Y \) is a regular space and a net \( \{ F_j : j \in J \} \) is lower strongly quasi-uniformly convergent to \( F \), then \( F \in (\tau_Y^-) \lim F_j \).

**Proof.** Let \( x_0 \in X \) and \( V^- \) be a set from subbase \( \mathcal{P} \) of lower Vietoris topology. It means that \( V \) is an open subset of \( Y \) such that \( V \cap F(x_0) \neq \emptyset \). Let us take an element \( y \) from this intersection. Since \( y \in V \) and \( Y \) is a regular space,
then there exists an open set \( W \) satisfying condition \( y \in W \subset \text{cl}(W) \subset V \). The family \( A := \{V, Y \setminus \text{cl}(W)\} \) forms an open cover of the space \( Y \). From lower strong quasi-uniform convergence of the net \( \{F_j : j \in J\} \) to \( F \) there exists \( j_0 \in J \) such that for every \( j, j \geq j_0 \), there is a neighbourhood \( U \) of the point \( x_0 \) with the inclusion \( F(x) \subset \text{St}(F_j(x), A) \) for every \( x \in U \). Let \( j, j \geq j_0 \), be fixed. There exists a neighbourhood \( U \) of the point \( x_0 \) such that \( F(x) \subset \text{St}(F_j(x), A) \) for every \( x \in U \). This condition is satisfied for \( x_0 \), too. Hence \( F(x_0) \subset \text{St}(F_j(x_0), A) \). Since \( y \in \text{St}(F_j(x_0), A) \), then there exists a set from the cover \( A \) containing \( y \).

We assumed that \( y \in W \), so \( y \notin Y \setminus \text{cl}(W) \). Hence \( y \in V \) and \( V \cap F_j(x_0) \neq \emptyset \) for \( j, j \geq j_0 \). In consequence, \( F \in (\tau_Y^-) \lim F_j \).

**Definition 8.** ([5]) Let \((X, \tau_X), (Y, \tau_Y)\) be topological spaces. A multivalued map \( F : X \to Y \) is said to be upper (respectively: lower) semi-continuous at the point \( x_0 \in X \) if for every open set \( V \) of \( Y \) such that \( F(x_0) \subset V \ [F(x_0)] \cap V \neq \emptyset \) there exists a neighbourhood \( U \) of the point \( x_0 \) with the property \( F(x) \subset V \ [F(x)] \cap V \neq \emptyset \) for every point \( x \) of the set \( U \).

The symbol \( C^+(F) \) (respectively: \( C^-(F) \) ) denotes the set of all points, in which the multivalued map \( F \) is upper (lower) semi-continuous.

**Theorem 3.** Let \((X, \tau_X)\) be a topological space. If \((Y, \tau_Y)\) is a regular space, a net \( \{F_j : j \in J\} \) of multivalued maps is \( \tau_Y^- \)-pointwise and upper strongly quasi-uniformly convergent to a multivalued map \( F \), then the inclusion

\[
\bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j) \subset C^-(F)
\]

holds.

**Proof.** Let \( x_0 \in \bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j) \). Therefore for every \( i \in J \) there exists \( j, j \geq i \), such that

\[
(1) \quad x_0 \in C^-(F_j).
\]

Let \( V \) be an open subset of \( Y \) such that \( F(x_0) \cap V \neq \emptyset \). Let us assume that \( y \in F(x_0) \cap V \). From the regularity of \( Y \) we can find an open subset \( W \) of \( Y \) with properties \( y \in W \subset \text{cl}(W) \subset V \). Moreover \( F \in (\tau_Y^-) \lim F_j \), so there exists \( j_0 \in J \) such that \( F_j(x_0) \cap W \neq \emptyset \) for every \( j, j \geq j_0 \). The
family $\mathcal{A} := \{V, Y \setminus \text{cl}(W)\}$ is an open cover of $Y$. The upper strong quasi-uniform convergence to $F$ implies the existence $j_1 \in J$ such that for every $j$, $j \geq j_1$ there exists the neighbourhood $U_1$ of $x_0$ fulfilling the inclusion $F_j(x) \subset \text{St}(F(x), \mathcal{A})$ if $x \in U_1$. We can choose $j_2$ such that $j_2 \geq j_0$ and $j_2 \geq j_1$. In view of condition (1) we will find an index $j$, $j \geq j_2$, such that $x_0 \in C^-(F_j)$. Then there exists a neighbourhood $U_2$ of $x_0$ such that $F_j(x) \cap W \neq \emptyset$ for every $x \in U_2$. Let us consider $U = U_1 \cap U_2$. The set $U$ is a neighbourhood of $x_0$ satisfying the conditions

\begin{align}
(2) & \quad F_j(x) \subset \text{St}(F(x), \mathcal{A}) \\
(3) & \quad F_j(x) \cap W \neq \emptyset
\end{align}

for every $x \in U$.

We will prove that $F(x) \cap W \neq \emptyset$ for each point $x$ of the set $U$. On the contrary, suppose that $F(x_1) \cap V = \emptyset$ for some $x_1 \in U$. It means that

$$F(x_1) \subset Y \setminus V \subset Y \setminus \text{cl}(W)$$

$$\text{St}(F(x_1), \mathcal{A}) = Y \setminus \text{cl}(W).$$

From (2) we obtain

$$F_j(x_1) \subset \text{St}(F(x_1), \mathcal{A}) = Y \setminus \text{cl}(W) \subset Y \setminus W.$$

Therefore $F_j(x_1) \cap W = \emptyset$, which is contrary to (3). In this way, we have proved that for every open set $V$ such that $F(x_0) \cap V \neq \emptyset$ there is a neighbourhood $U$ of $x_0$ with condition $F(x) \cap V \neq \emptyset$ for every $x \in U$. It means that the map $F$ is lower semi-continuous at the point $x_0$ and $x_0 \in C^-(F)$. In consequence

$$\bigcap_{i \in J \atop j \geq i} C^-(F_j) \subset C^-(F),$$

which finishes the proof. $\square$

**Corollary 1.** Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces, and $Y$ be a regular space, a net $\{F_j : j \in J\}$ of multivalued maps $F_j : X \rightarrow Y$ be $\tau_Y^-$-pointwise and upper strongly quasi-uniformly convergent to a map $F$. If $F_j$ are lower semi-continuous for every $j \in J$, then $F$ is lower semi-continuous.
Proof. The assumptions of Theorem 3 are satisfied, so we have the inclusion
\[ \bigcap_{i \in J} \bigcup_{j \geq i} C^-(F_j) \subset C^-(F). \]
Since \( F_j \) are lower semi-continuous, so \( C^-(F_j) = X \) for every \( j \in J \). Therefore \( X \subset C^-(F) \). In result \( C^-(F) = X \) and \( F \) is the lower semi-continuous multivalued map. \( \square \)

**Theorem 4.** Let \((X, \tau_X)\) be a topological space, and \((Y, \tau_Y)\) be a regular space and \( F : X \to Y \) be a multivalued map with \( \alpha \)-paracompact values. If a net \( \{F_j : j \in J\} \) of multivalued maps is \( \tau_Y^+ \)-pointwise and lower strongly quasi-uniformly convergent to \( F \), then
\[ \bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F). \]

Proof. Assume that
\[ x_0 \in \bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j). \]
Let \( V \) be an open set with the property \( F(x_0) \subset V \).

There exists an open set \( W \) such that
\[ F(x_0) \subset W \subset \text{cl}(W) \subset V. \]
because of \( F(x_0) \) is \( \alpha \)-paracompact set in the regular space \( Y \) and Lemma 2.

Then the family \( \mathcal{A} := \{V, Y \setminus \text{cl}(W)\} \) is an open cover of the space \( Y \). The net \( \{F_j : j \in J\} \) is \( \tau_Y^+ \)-pointwise convergent to the map \( F \). Therefore there exists an index \( j_1 \in J \) such that \( F_j(x_0) \subset W \) for every \( j, j \geq j_1 \). From lower strong quasi-uniform convergence of the net \( \{F_j : j \in J\} \) to \( F \) follows that there is \( j_2 \in J \) such that for every \( j, j \geq j_2 \) there exists a neighbourhood \( U_1 \) of the point \( x_0 \) such that \( F(x) \subset \text{St}(F_j(x), \mathcal{A}) \) for every \( x \in U_1 \). Let \( j_0 \) be such that \( j_0 \geq j_1 \) and \( j_0 \geq j_2 \). By (4) there exists \( j, j \geq j_0 \), such that \( x_0 \in C^+(F_j) \). Since the map \( F \) is upper semi-continuous at the point \( x_0 \) and \( F_j(x_0) \subset W \) we can choose a neighbourhood \( U_2 \) of the point \( x_0 \) such that \( F_j(x) \subset W \) for every \( x \in U_2 \). Then the set \( U := U_1 \cap U_2 \) is open, \( x_0 \in U \) and
\[ F(x) \subset \text{St}(F_j(x), \mathcal{A}) \text{ and } F_j(x) \subset W \]
Therefore
\[ \text{St}(F_j(x), \mathcal{A}) = V. \]
and in consequence $F(x) \subset V$ for every $x \in U$. We obtained that for every open set $V$ such that $F(x_0) \subset V$ there exists a neighbourhood $U$ of $x_0$ such that $F(x) \subset V$ for every $x \in U$. It means that $x_0 \in C^+(F)$. Then

$$\bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F).$$

\[\square\]

**Corollary 2.** Let $(X, \tau_X)$ be a topological space, $(Y, \tau_Y)$ be a regular space and a map $F : X \to Y$ has $\alpha$-paracompact values. If multifunctions $F_j : X \to Y$ are upper semi-continuous for $j \in J$ and the net $\{F_j : j \in J\}$ is $\tau_Y^+$-pointwise and lower strongly quasi-uniformly convergent to the map $F$, then $F$ is upper semi-continuous.

**Proof.** All assumptions of Theorem 4 are satisfied. Hence we have

$$X = \bigcap_{i \in J} \bigcup_{j \geq i} C^+(F_j) \subset C^+(F) \subset X.$$ 

Thus $C^+(F) = X$ and it is equivalent to upper semi-continuity of the map $F$ on the space $X$. \[\square\]

**Definition 9.** ([3]) Let $X$ and $Y$ be topological spaces. A net $\{F_j : j \in J\}$ of multivalued maps from $X$ to $Y$ is called frequently upper (lower) semi-continuous at a point $x_0 \in X$ if for every $j_0 \in J$ there exists $j \in J$, $j \geq j_0$, such that the multivalued map $F_j$ is upper (lower) semi-continuous at the point $x_0$.

**Theorem 5.** Let $(X, \tau_X)$ be a topological space, $(Y, \tau_Y)$ be a regular space. Let $F_j, F$ be multivalued maps from $X$ to $Y$ for every $j \in J$. Assume that there exists $x_0 \in X$ such that $F(x_0)$ is an $\alpha$–paracompact set. If the net $\{F_j : j \in J\}$ is upper and lower strongly quasi-uniformly convergent to $F$ and it is frequently upper semi-continuous at the point $x_0$, then the map $F$ is upper semi-continuous at $x_0$.

**Proof.** Let $V$ be an open subset of the regular space $Y$ such that $F(x_0) \subset V$. The set $F(x_0)$ is $\alpha$–paracompact, so by virtue of Lemma 2 there exist open sets $G, H$ satisfying inclusions

$$F(x_0) \subset G \subset \text{cl}(G) \subset H \subset \text{cl}(H) \subset V.$$
Let us consider open covers $\mathcal{A} := \{V, Y \setminus \text{cl}(H)\}$ and $\mathcal{B} := \{H, Y \setminus \text{cl}(G)\}$ of the space $Y$. Assumptions about upper and lower strong quasi-uniform convergence of the net $\{F_j : j \in J\}$ to $F$ imply conditions:

\begin{align*}
(5) & \quad \exists j_0 \in J \forall j \geq j_0 \exists U_1 - \text{neighbourhood of } x_0 \forall x \in U_1 F_j(x) \subset \text{St}(F(x), \mathcal{B}) \\
(6) & \quad \exists j_1 \in J \forall j \geq j_1 \exists U_2 - \text{neighbourhood of } x_0 \forall x \in U_2 F(x) \subset \text{St}(F_j(x), \mathcal{A}).
\end{align*}

The set $J$ is directed, so there exists $j_2$ such that $j_2 \geq j_0$ and $j_2 \geq j_1$. Moreover, the net $\{F_j : j \in J\}$ is frequently upper semi-continuous at $x_0$. Therefore, there exists an index $j$, $j \geq j_2$ such that a map $F_j$ is upper semi-continuous at $x_0$. Then conditions (5) and (6) are satisfying for $j$. It means that there exists a neighbourhood $G_1$ of $x_0$ such that $F_j(x) \subset \text{St}(F(x), \mathcal{B})$ for each $x \in G_1$ and there exists a neighbourhood $G_2$ of the point $x_0$ such that $F(x) \subset \text{St}(F_j(x), \mathcal{A})$ for each $x \in G_2$. If $G = G_1 \cap G_2$, then $G$ is a neighbourhood of $x_0$. In particular, $F_j(x_0) \subset \text{St}(F(x_0), \mathcal{B}) = H$. The map $F_j$ is upper semi-continuous at $x_0$. Therefore there exists a neighbourhood $U_3, U_3 \subset U$, of the point $x_0$, such that $F_j(x) \subset H$ for every $x \in U_3$. For $x \in U_3$ we have $F(x) \subset \text{St}(F_j(x), \mathcal{A}) = V$.

Thus we proved that for every open set $V$ such that $F(x_0) \subset V$ there is a neighbourhood $U_3$ of the point $x_0$ with the property $F(x) \subset V$ for every $x \in U_3$. It means that $F$ is upper semi-continuous at the point $x_0$. $\square$

**Theorem 6.** Let $(X, \tau_X)$ be a topological space, $(Y, \tau_Y)$ be a regular space. Assume that $F_j, F$ are multivalued maps from $X$ to $Y$ for every $j \in J$. Let $x_0 \in X$. If the net $\{F_j : j \in J\}$ is frequently lower semi-continuous at the point $x_0$ and it converges upper and lower strongly quasi-uniformly to $F$, then the map $F$ is lower semi-continuous at $x_0$.

**Proof.** Let $V$ be an open subset of the space $Y$ such that $F(x_0) \cap V \neq \emptyset$. Let $z \in F(x_0) \cap V$. Since the space $Y$ is regular, $z \in V$ and $V$ is the open set, so there exist open sets $G, W$ such that

$$z \in G \subset \text{cl}(G) \subset W \subset \text{cl}(W) \subset V.$$

We will consider the following open covers of the space $Y$:

$$\mathcal{A} := \{W, Y \setminus \text{cl}(G)\}, \quad \mathcal{B} := \{V, Y \setminus \text{cl}(W)\}.$$
Conditions of upper and lower strong quasi-uniform convergence imply the facts:

\[(7)\]  \[\exists_{j_1 \in J} \forall_{j_2 \geq j_1} \exists U_1 \text{-neighbourhood of } x_0 \forall x \in U_1 F_j(x) \subset \text{St}(F(x), \mathcal{B})\]

\[(8)\]  \[\exists_{j_2 \in J} \forall_{j_3 \geq j_2} \exists U_2 \text{-neighbourhood of } x_0 \forall x \in U_2 F(x) \subset \text{St}(F_j(x), \mathcal{A}).\]

Since \(J\) is a directed set, there exists \(j_3 \in J\) such that \(j_3 \geq j_1\) and \(j_3 \geq j_2\). The net \(\{F_j : j \in J\}\) is frequently lower semi-continuous at the point \(x_0\). Therefore, there is an index \(j, j \geq j_3\) such that a map \(F_j\) is lower semi-continuous at the point \(x_0\). From conditions \((7)\) and \((8)\) we obtain

\[(9)\]  \[\exists G_1 \text{-neighbourhood of } x_0 \forall x \in G_1 F_j(x) \subset \text{St}(F(x), \mathcal{B})\]

\[(10)\]  \[\exists G_2 \text{-neighbourhood of } x_0 \forall x \in G_2 F(x) \subset \text{St}(F_j(x), \mathcal{A}).\]

Let us consider the set \(U_3 := U_1 \cap U_2\), which is a neighbourhood of \(x_0\). Now, we will prove that \(F_j(x_0) \cap W \neq \emptyset\). Suppose that \(F_j(x_0) \cap W = \emptyset\). It implies inclusions \(F_j(x_0) \subset Y \setminus W \subset Y \setminus \text{cl}(G)\). From \((10)\) we infer that

\[F(x_0) \subset \text{St}(F_j(x_0), \mathcal{A}) = Y \setminus \text{cl}(V) \subset Y \setminus V,\]

which contradicts to \(z \in F(x_0) \cap V\). Thus \(F(x_0) \cap V \neq \emptyset\).

In consequence we showed that \(F_j(x_0) \cap W \neq \emptyset\). The map \(F_j\) is lower semi-continuous at \(x_0\), so there exists a neighbourhood \(U \subset U_3\) such that

\[(11)\]  \[F_j(x) \cap W \neq \emptyset\]

for every \(x \in U\).

By \((9)\) we obtain \(F_j(x) \subset \text{St}(F(x), \mathcal{B})\). Then \(F(x) \cap V \neq \emptyset\) for each \(x \in U\). Indeed, if \(F(x') \cap V = \emptyset\) for any \(x' \in U\), then

\[F_j(x') \subset \text{St}(F(x'), \mathcal{B}) = Y \setminus \text{cl}(W) \subset Y \setminus W,\]

which contradicts to \((11)\). Thus we proved the following condition: for every open set \(V\) such that \(F(x_0) \cap V \neq \emptyset\), there exists a neighbourhood \(U\) of the point \(x_0\) such that \(F(x) \cap V \neq \emptyset\) for every \(x \in U\).

It means that \(F\) is lower semi-continuous at the point \(x_0\).

\[\square\]

References


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UNIFORML Y BOUNDED SET-VALUED COMPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF BOUNDED V ARIA TION IN THE SENSE OF SCHRAMM

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Abstract

We prove that, under some general assumptions, the one-sided regularizations of the generator of any uniformly bounded set-valued composition operator, acting in the spaces of functions of bounded variation in the sense of Schramm with nonempty bounded closed and convex values is an affine function. As a special case, we obtain an earlier result ([15]).

1. Introduction

Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be real normed spaces, $C$ be a convex cone in $X$ and $I = [a, b] \subset \mathbb{R}$ be an interval. Let $\text{clb}(Y)$ be the family of all nonempty bounded, closed and convex subsets of $Y$. We consider the Nemytskij operator, i.e., for a function $F : I \to C$ the composition operator is defined by

$$(HF)(t) = h(t, F(t)),$$

where $h : I \times C \to \text{clb}(Y)$ is a given set-valued function. It is shown that if operator $H$ maps the space $\Phi BV(I; C)$ of functions of bounded $\Phi$-variation in the sense of Schramm into the space $\text{BS}_{\Psi}(I; \text{clb}(Y))$ of closed bounded convex valued functions of bounded $\Psi$-variation in the sense of Schramm and

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is uniformly bounded, then the one-sided regularizations \( h^- \) and \( h^+ \) of \( h \), with respect to the first variable, are affine with respect to the second variable, i.e., Matkowski’s representation holds. In particular,

\[
h^-(t,x) = A(t)x^* + B(t), \quad \text{for } t \in I, \ x \in C,
\]

for some function \( A : I \to \mathcal{L}(C,\text{clb}(Y)) \) and \( B \in \text{BS}_\Psi(I;\text{clb}(Y)) \), where \( \mathcal{L}(C,\text{clb}(Y)) \) stands for the space of all linear mappings acting from \( C \) into \( \text{clb}(Y) \). This extends the main result of the paper [15].

2. Preliminaries

Let \( \mathcal{F} \) be the set of all convex functions \( \varphi : [0, \infty) \to [0, \infty) \) vanishing at zero only (and, hence, continuous on \( [0, \infty) \) and strictly increasing).

A sequence \( \Phi = (\phi_i)_{i=1}^\infty \) of functions from \( \mathcal{F} \) satisfying the following two conditions:

(i) \( \phi_{n+1}(t) \leq \phi_n(t) \) for all \( t > 0 \) and \( n \in \mathbb{N} \),

(ii) \( \sum_{n=1}^\infty \phi_n(t) \) diverges for all \( t > 0 \),

is said to be a \( \Phi \)-sequence.

Let \( I \subset \mathbb{R} \) be an interval. For a set \( X \) we denote by \( X^I \) the set of all functions which map \( I \) into \( X \).

If \( I_n = [a_n, b_n] \) is a subinterval of the interval \( I \ (n = 1, 2, \ldots) \), then we write \( f(I_n) = f(b_n) - f(a_n) \).

**Definition 1.** [14] Let \( \Phi = (\phi_i)_{i=1}^\infty \) be a \( \Phi \)-sequence and \( (X,| \cdot |_X) \) be a real normed space. A function \( f \in X^I \) is said to have bounded \( \Phi \)-variation in the sense of Schramm in \( I \), if

\[
v_\Phi(f) = v_\Phi(f,I) := \sup \sum_{n=1}^m \phi_n(|f(I_n)|) < \infty,
\]

where the supremum is taken over all \( m \in \mathbb{N} \) and all non-ordered collections of non-overlapping intervals \( I_n = [a_n, b_n], \ n = 1, \ldots, m \).

It is known that for all \( a, b, c \in I \), such that \( a \leq c \leq b \) we have

\[
v_\Phi(f, [a, c]) \leq v_\Phi(f, [a, b])
\]

(that is, \( v_\Phi \) is increasing with respect to the interval) and

\[
v_\Phi(f, [a, c]) + v_\Phi(f, [c, b]) \leq v_\Phi(f, [a, b]).
\]
We will denote by $V_{\Phi}(I, X)$ the set of all bounded $\Phi$-variation functions $f \in X^I$ in the Schramm sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak-Orlicz proved the following statement: this class of functions is a vector space if and only if $\varphi$ satisfies the $\delta_2$ condition [11].

By $\Phi BV(I, X)$ we will denote the linear space of all functions $f \in X^I$ such that $v_{\Phi}(\lambda f) < \infty$ for some positive constant $\lambda$.

In the linear space $\Phi BV(I, X)$, the function $\| \cdot \|_{\Phi}$ defined by

$$\| f \|_{\Phi} := |f(a)| + p_{\Phi}(f), \quad f \in \Phi BV(I, X),$$

where

$$p_{\Phi}(f) := p_{\Phi}(f, I) = \inf \{ \epsilon > 0 : v_{\Phi}(f/\epsilon) \leq 1 \}, \quad (2)$$

is a norm (see for instance [11]).

The linear normed space $(BV_{\Phi}(I, \mathbb{R}), \| \cdot \|_{\Phi})$ was studied by Schramm (Theorem 2.3 [14]).

The functional $p_{\varphi}(\cdot)$ defined by (2) is called the Luxemburg-Nakano-Orlicz seminorm [5, 19, 20].

It is worth mentioning that the symbol $\Phi BV(I, C)$ stands for the set of all functions $f \in \Phi BV(I, X)$ such that $f : I \to C$ and $C$ is a subset of $X$.

Let $(Y, \cdot, \cdot_Y)$ be a normed real vector space.

The family of all nonempty bounded closed convex subsets of $Y$ equipped with the Hausdorff metric $D$ generated by the norm in $Y$:

$$D(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} |a - b|_Y, \sup_{b \in B} \inf_{a \in A} |a - b|_Y \right\}, \quad A, B \in \text{clb}(Y).$$

is denoted by $\text{clb}(Y)$.

Given $A, B \in \text{clb}(Y)$, we put $A + B := \{a + b : a \in A, \ b \in B\}$ and we introduce the operation $+$ in $\text{clb}(Y)$ defined as follows:

$$A + B = \text{cl}(A + B),$$

where cl stands for the closure in $Y$. The class $\text{clb}(Y)$ with the operation $+$ is an Abelian semigroup, with $\{0\}$ as the zero element, which satisfies the cancelation law. Moreover, we can multiply elements of $\text{clb}(Y)$ by nonnegative numbers and, for all $A, B \in \text{clb}(Y)$ and $\lambda, \mu \geq 0$, the following conditions hold:

$$1 \cdot A = A, \lambda(\mu A) = (\lambda \mu)A, \lambda(A + B) = \lambda A + \lambda B, \lambda(\lambda + \mu)A = \lambda A + \mu A.$$
In view of [4, lemma 3]

\[ D(A \ast B, A \ast C) = D(A + B, A + C) = D(B, C); \quad A, B, C \in \text{clb}(Y), \]  

(\text{clb}(Y), D, +, \cdot) is an abstract convex cone; this cone is complete provided \( Y \) is a Banach space.

**Definition 2.** Let \( \Phi = (\phi_n)_{n=1}^{\infty} \) be a \( \Phi \)-sequence and \( F : I \rightarrow \text{clb}(X) \). We say that \( F \) has bounded \( \Phi \)-variation in the Schramm sense if

\[ w_\Phi(F) := \sup \sum_{n=1}^{m} \Phi_n(D(F(t_n), F(t_{n-1}))) < \infty, \]  

where the least upper bound is taken over all \( m \in \mathbb{N} \) and all collections of non-overlapping intervals \( I_n = [a_n, b_n] \subset I, i = 1, \ldots, m. \)

From now on, let

\[ BS_\Phi(I, \text{clb}(X)) := \left\{ F \in \text{clb}(X)^I : w_\Phi(\lambda F) < \infty \text{ for some } \lambda > 0 \right\}. \]  

For \( F_1, F_2 \in BS_\Phi(I, \text{clb}(X)) \) put

\[ D_\Phi(F_1, F_2) := D(F_1(a), F_2(a)) + p_\Phi(F_1, F_2) \]  

where

\[ p_\Phi(F_1, F_2) := \inf \left\{ \epsilon > 0 : S_\epsilon(F_1, F_2) \leq 1 \right\} \]  

and

\[ S_\epsilon(F_1, F_2) := \sup \sum_{n=1}^{m} \phi_n \left( \frac{1}{\epsilon} \lambda \left( F_1(t_n) + \epsilon F_2(t_{n-1}) + F_2(t_n) + \epsilon F_1(t_{n-1}) \right) \right), \]  

where the least upper bound is taken over the same collection \( ([a_n, b_n])_{n=1}^{m} \) as in Definition 2. Then \( (BS_\Phi(I, \text{clb}(X)), D_\Phi) \) is a metric space, and it is complete if \( X \) is a Banach space [18, Lemma 5.4].

Taking into account [17, Theorem 3.8 (d)] and [18, condition 5.6] we get the following:

**Lemma 3.** Let \( \Phi = (\phi_n)_{n=1}^{\infty} \) be a \( \Phi \)-sequence and \( F_1, F_2 \in BS_\Phi(I, \text{clb}(X)) \). Then, for \( \lambda > 0 \), \( S_\lambda(F_1, F_2) \leq 1 \) if and only if \( p_\Phi(F_1, F_2) \leq \lambda. \)

Let \( (X, | \cdot |_X), (Y, | \cdot |_Y) \) be two real normed spaces. A subset \( C \subset Y \) is said to be a convex cone if \( \lambda C \subset C \) for all \( \lambda \geq 0 \) and \( C + C \subset C \). It is obvious
that $0 \in C$. Given a set-valued function $h : I \times C \rightarrow \text{clb}(Y)$ we consider the composition operator $H : C^I \rightarrow \text{clb}(Y)^I$ generated by $h$, i.e.,

$$(Hf)(t) := h(t, f(t)), \quad f \in C^I, \quad t \in I.$$ 

A set-valued function $F : C \rightarrow \text{clb}(Y)$ is said to be *additive, if

$$F(x + y) = F(x) + F(y),$$

and *Jensen if

$$2F\left(\frac{x + y}{2}\right) = F(x) + F(y),$$

for all $x, y \in C$.

We will need the following

**Lemma 4** (21, Cor. 4). Let $C$ be a convex cone in a real linear space and let $(Y, \|\cdot\|_Y)$ be a Banach space. A set-valued function $F : C \rightarrow \text{clb}(Y)$ is *Jensen if and only if there exists an *additive set-valued function $A : C \rightarrow \text{clb}(Y)$ and a set $B \in \text{clb}(Y)$ such that

$$F(x) = A(x) + B,$$

for all $x \in C$.

For the normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ by $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$, briefly $\mathcal{L}(X, Y)$, we denote the normed space of all additive and continuous mappings $A \in Y^X$.

Let $C$ be a convex cone in a real normed space $(X, \|\cdot\|_X)$. From now on, let the set $\mathcal{L}(C, \text{clb}(Y))$ consists of all set-valued functions $A : C \rightarrow \text{clb}(Y)$ which are *additive and continuous (so positively homogeneous), i.e.,

$$\mathcal{L}(C, \text{clb}(Y)) = \{A \in \text{clb}(Y)^C : A \text{ is } \ast \text{additive and continuous}\}.$$

The set $\mathcal{L}(C, \text{clb}(Y))$ can be equipped with the metric defined by

$$d_{\mathcal{L}(C,\text{clb}(Y))}(A, B) := \sup_{y \in C \setminus \{0\}} \frac{d(A(y), B(y))}{\|y\|_Y}.$$ 

**Theorem 5.** Let $(X, \|\cdot\|_X)$ be a real normed space, $(Y, \|\cdot\|_Y)$ a real Banach space, $C$ a convex cone in $X$ and let $\Phi, \Psi \in \mathcal{F}$. Suppose that the set-valued function is continuous with respect to the second variable. If the composition operator $H$ generated by a set-valued function $h : I \times C \rightarrow \text{clb}(Y)$ maps $\Phi BV(I, C)$ into $BS_\Psi(I, \text{clb}(Y))$ and satisfies the inequality,

$$D_\Phi(H(f_1), H(f_2)) \leq \omega(||f_1 - f_2||), \quad f_1, f_2 \in \Phi BV(I, C),$$

...
for some function $\omega : [0, \infty) \to [0, \infty)$, then the left regularization of $h$, i.e. the function $h^- : I \times X \to Y$ defined by
\[
h^-(t, x) := \lim_{s \downarrow t} h(s, x), \quad t \in I, \ x \in C,
\]
exists and
\[
h^-(t, x) = A(t)x + B(t), \quad t \in I, \ x \in C,
\]
for some $A : I \to \mathcal{A}(X, \text{clb}(Y))$ and $B : I \to \text{clb}(Y)$. Moreover, if $0 \in C$, then $B \in BS_\Phi(I, \text{clb}(Y))$ and the linear set-valued function $A(t)$ is continuous.

**Proof.** For every $x \in C$ the constant function $I \ni t \mapsto x$ belongs to $\Phi BV(I, C)$. Since $H$ maps $\Phi BV(I, C)$ into $BS_\Phi(I, \text{clb}(Y))$ for every $x \in C$ the function $I \ni t \mapsto h(t, x)$ belongs to $BS_\Phi(I, \text{clb}(Y))$. The completeness of $\text{clb}(Y)$ with respect to the Hausdorff metric [18, Lemma 6.12] implies the existence of the left regularization $h^-$ of $h$.

Function $H$ is uniformly continuous on $\Phi BV(I, C)$. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be the modulus of continuity of $H$, that is
\[
\omega(\rho) := \sup \left\{ D_\Phi(H(f_1), H(f_2)) : \|f_1 - f_2\|_\Phi \leq \rho; \ f_1, f_2 \in \Phi BV(I, C) \right\}
\]
if $\rho > 0$. Hence we get
\[
D_\Phi(H(f_1), H(f_2)) \leq \omega(\|f_1 - f_2\|_\Phi) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{9}
\]
From the definition of the metric $D_\Phi$ and (9) we obtain
\[
p_\Phi(H(f_1); H(f_2)) \leq \omega(\|f_1 - f_2\|_\Phi) \quad \text{for} \quad f_1, f_2 \in \Phi BV(I, C). \tag{10}
\]
From Lemma 3 if $\omega(\|f_1 - f_2\|_\Phi) > 0$ the inequality (10) is equivalent to
\[
S_{\omega(\|f_1 - f_2\|_\Phi)}(H(f_1), H(f_2)) \leq 1, \quad f_1, f_2 \in \Phi BV(I, C). \tag{11}
\]
Therefore, for any
\[
a < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m = b,
\]
where $\alpha_i, \beta_i \in I$, $i \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, the definitions of the operator $H$ and the functional $S_\varepsilon$ imply
\[
\sum_{i=1}^{\infty} \psi_i \left( \frac{D(h(\beta_i, f_1(\beta_i))^{\ast} + h(\alpha_i, f_2(\alpha_i)); h(\beta_i, f_2(\beta_i))^{\ast} + h(\alpha_i, f_1(\alpha_i)))}{\omega(\|f_1 - f_2\|_\Phi)} \right) \leq 1. \tag{12}
\]
For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, we define functions $\eta_{\alpha, \beta} : \mathbb{R} \to [0, 1]$ by
\( \eta_{\alpha, \beta}(t) := \begin{cases} 0 & \text{if } t \leq \alpha \\ \frac{t - \alpha}{\beta - \alpha} & \text{if } \alpha \leq t \leq \beta \\ 1 & \text{if } \beta \leq t. \end{cases} \) \tag{13}

Let us fix \( t \in I \). For arbitrary finite sequence \( \alpha < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m < t \) and \( x_1, x_2 \in C, x_1 \neq x_2 \), the functions \( f_1, f_2 : I \rightarrow X \) defined by

\[ f_i(\tau) := \frac{1}{2} \left[ \eta_{\alpha_i, \beta_i}(\tau)(x_1 - x_2) + x_\ell + x_2 \right], \quad \tau \in I, \ell = 1, 2 \tag{14} \]

belong to the space \( \Phi BV(I, C) \). From (14) we have

\[ f_1(\tau) - f_2(\tau) = \frac{x_1 - x_2}{2}, \quad \tau \in I, \]

therefore

\[ \|f_1 - f_2\|_\Phi = \left| \frac{x_1 - x_2}{2} \right|; \]

moreover

\[ f_1(\beta_i) = x_1; \quad f_2(\beta_i) = \frac{x_1 + x_2}{2}; \quad f_1(\alpha_i) = \frac{x_1 + x_2}{2}; \quad f_2(\alpha_i) = x_2. \]

Applying (12) we hence get

\[ \sum_{i=1}^{\infty} \psi_i \left( \frac{D \left( h(\beta_i, x_1) + h(\alpha_i, x_2); h \left( \alpha_i, \frac{x_1 + x_2}{2} \right) \right)}{\omega \left( \frac{x_1 - x_2}{2} \right)} \right) \leq 1. \tag{15} \]

For a fixed positive integer \( m \) we have:

\[ \sum_{i=1}^{m} \psi_i \left( \frac{D \left( h(\beta_i, x_1) + h(\alpha_i, x_2); h \left( \alpha_i, \frac{x_1 + x_2}{2} \right) \right)}{\omega \left( \frac{x_1 - x_2}{2} \right)} \right) \leq 1. \tag{16} \]

From the continuity of \( \psi_i \) passing to the limit in (16) when \( \alpha_1 \uparrow t \) we infer that

\[ \sum_{i=1}^{m} \psi_i \left( \frac{D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-(t, \frac{x_1 + x_2}{2}) \right)}{\omega \left( \frac{x_1 - x_2}{2} \right)} \right) \leq 1. \]
Hence,
\[ \sum_{i=1}^{\infty} \psi_i \left( \frac{D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-(t, \frac{x_1 + x_2}{2}) \right)}{\omega \left( \frac{|x_1 - x_2|}{2} \right)} \right) \leq 1, \]
and by (ii)
\[ D \left( h^-(t, x_1) + h^-(t, x_2); 2h^-(t, \frac{x_1 + x_2}{2}) \right) = 0. \]
Therefore
\[ h^-(t, \frac{x_1 + x_2}{2}) = \frac{h^-(t, x_1) + h^-(t, x_2)}{2} \]
for all \( t \in I \) and all \( x_1, x_2 \in C \).

Thus for each \( r \in I^- \) the function \( h^-(r, \cdot) \) satisfies the \(^*\)Jensen functional equation in \( C \) and by Lemma 4. for every \( t \in I \) there exist an \(^*\)additive set-valued function \( A(t) : C \to \text{clb}(Y) \) and a set \( B(t) \in \text{clb}(Y) \) such that
\[ h^-(t, x) = A(t)x^* + B(t) \quad \text{for} \quad x \in C, \ t \in I. \]  

To show that \( A(t) \) is continuous for any \( t \in I \), let us fix \( x, \overline{x} \in C \). By (3) and (17) we have
\[ D(A(t)x, A(t)\overline{x}) = D(A(t)x^* + B(t), A(t)\overline{x}^* + B(t)) = D(h(t, x), h(t, \overline{x})). \]
Hence, the continuity of \( h \) with respect to the second variable implies the continuity of \( A(t) \) and, consequently, being \(^*\)additive, \( A(t) \in \mathcal{L}(C, \text{clb}(Y)) \) for every \( t \in I \). To prove that \( B \in BW_\psi(I, \text{clb}(Y)) \) let us note that the \(^*\)additivity of \( A(t) \) implies \( A(t)0 = \{0\} \). Therefore, putting \( x = 0 \) in (17), we get
\[ h^-(t, 0) = B(t), \quad t \in I, \]
which gives the required claim.

**Remark 6.** The counterpart of Theorem 5. for the right regularization \( h^+ \) of \( h \) defined by
\[ h^+(t, x) := \lim_{s \uparrow t} h(s, x); \quad t \in I, \]
is also true.

**Remark 7.** Taking \( \psi_n(t) = \psi(t), \ (t \geq 0) \), we obtain the main result of [1].
3. Uniformly bounded set-valued composition operators

In this section we present the definition given recently by Matkowski [6] for uniformly bounded composition operators.

**Definition 8.** Let $X$ and $Y$ be two metric (or normed) spaces. We say that a mapping $H : X \to Y$ is uniformly bounded if for any $t > 0$ there is a real number $\gamma(t)$ such that for any nonempty set $B \subset X$ we have

$$\text{diam } B \leq t \implies \text{diam } H(B) \leq \gamma(t).$$

**Remark 9.** Obviously, every uniformly continuous operator or Lipschitzian operator is uniformly bounded. Note that, under the assumptions of this definition, every bounded operator is uniformly bounded.

The following theorem represents our main result.

**Theorem 10.** Let $(X, | \cdot |_X)$ be a real normed space, $(Y, | \cdot |_Y)$ a real Banach space, $C$ a convex cone in $X$ and let $\Phi, \Psi \in \mathcal{F}$. If the composition operator $H$ generated by a set-valued function $h : I \times C \to \text{clb}(Y)$ maps $\Phi BV(I, C)$ into $BS\Psi(I, \text{clb}(Y))$ and is uniformly bounded, then the left regularization of $h$, i.e., the function $h^- : I \times X \to Y$ defined by

$$h^-(t, x) := \lim_{s \uparrow t} h(s, x), \quad t \in I, \ x \in C,$$

exists and

$$h^-(t, x) = A(t)x + B(t), \quad t \in I, \ x \in C,$$

for some $A : I \to A(X, \text{clb}(Y))$ and $B : I \to \text{clb}(Y)$.

**Proof.** Let $t \geq 0$ and $f_1, f_2 \in \Phi BV(I; C)$ fulfill the condition

$$||f_1 - f_2||_\Phi \leq t.$$

Since $\text{diam } \{f_1, f_2\} \leq t$, by the uniform boundedness of $H$, we have

$$\text{diam } H(\{f_1, f_2\}) \leq \gamma(t),$$

that is

$$||H(f_1) - H(f_2)||_\Psi = \text{diam } H(\{f_1, f_2\}) \leq \gamma(||f_1 - f_2||_\Phi)$$

and the result follows from Theorem 5. \qed
Remark 11. If the function $\gamma : [0, \infty) \rightarrow [0, \infty)$ in the Definition 8. is right continuous at 0 and $\gamma(0) = 0$ (or if only $\gamma(0+) = 0$), then, clearly the uniform boundedness of the involved operator reduces to its uniform continuity. It follows that Theorem 10. improves the result of [15, Th. 2.1] where $H$ is assumed to be uniformly continuous.

Let us remark that the uniform boundedness of an operator (weaker than the usual boundedness) introduced and applied in [6] for the Nemytskij composition operators acting between spaces of Hölder functions in the single-valued case and then extended to the set-valued cases in [10] for the operators with convex and compact values.

Recall that the representation of Lipschitz continuous Nemytskij operators acting in the spaces of functions of bounded variation was first considered in [9] and then in [8] (in the single-valued case), and in [3] in the set-valued case. Let us add that A. Smajdor and W. Smajdor [2], extending the single-valued result of [7], initiated interesting and important study of the set-valued case.

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TWISTED GROUP ALGEBRAS OF SUR-TYPE
OFFINITE GROUPS OVER AN INTEGRAL DOMAIN
OF CHARACTERISTIC \( p \)

DARIUSZ KLEIN

Abstract

Let \( S \) be an integral domain of positive characteristic \( p \), which is not a field, \( S^* \) the unit group of \( S \), \( G \) a finite group, and \( S^\lambda G \) the twisted group algebra of the group \( G \) over \( S \) with a 2-cocycle \( \lambda \in \mathbb{Z}^2(G, S^*) \). Denote by \( \text{Ind}_m(S^\lambda G) \) the set of isomorphism classes of indecomposable \( S^\lambda G \)-modules of \( S \)-rank \( m \). We exhibit algebras \( S^\lambda G \) of SUR-type, in the sense that there exists a function \( f_\lambda : \mathbb{N} \to \mathbb{N} \) such that \( f_\lambda(n) \geq n \) and \( \text{Ind}_{f_\lambda(n)}(S^\lambda G) \) is an infinite set for every integer \( n > 1 \).

1. Introduction

Let \( p \geq 2 \) be a prime. Gudyvok [4] and Jamusz [8, 9] showed that if \( K \) is an infinite field of characteristic \( p \) and \( G \) is a non-cyclic \( p \)-group for which \( |G/G'| \neq 4 \), then \( \text{Ind}_n(KG) \) is an infinite set for every integer \( n > 1 \). Let \( G \) be a finite \( p \)-group of order \( |G| > 2 \), \( K \) a commutative local ring of characteristic \( p \), and \( \text{rad} K \neq 0 \). Gudyvok and Chukhray [5, 6] proved that if \( K := K/\text{rad} K \) is an infinite field or \( K \) is an integral domain, then \( \text{Ind}_n(KG) \) is infinite for every integer \( n > 1 \). In paper [7], jointly with Sygetij, they obtained a similar result in the case where \( G \) is a non-cyclic \( p \)-group, \( p \neq 2 \) and \( K \) is an infinite ring of characteristic \( p \) or \( K \) is an infinite field. The similar problem was studied in [2], [3] for twisted group algebras \( K^\lambda G \), where \( K \) is a field of characteristic \( p \) or a commutative local ring of characteristic \( p \).

In this paper we exhibit twisted group algebras \( S^\lambda G \) of SUR-type, where \( S \) is an integral domain of characteristic \( p \) and \( G \) is a finite group.
Let $K$ be a commutative ring of characteristic $p$, $K^*$ the unit group of $K$, $G$ a finite group, $e$ the identity element of $G$, $G_p$ a Sylow $p$-subgroup of $G$ and $G'_p$ the commutator subgroup of $G_p$. We suppose that $p$ divides $|G|$ and $G_p$ is a normal subgroup of $G$. The twisted group algebra of $G$ over $K$ with a 2-cocycle $\lambda \in H^2(G, K^*)$ is the free $K$-algebra $K\lambda G$ with a $K$-basis $\{u_g : g \in G\}$ satisfying $u_au_b = \lambda_{a,b}u_{ab}$ for all $a, b \in G$. The $K$-basis $\{u_g : g \in G\}$ is called canonical (corresponding to $\lambda$). By a $K\lambda G$-module we mean a finitely generated left $K\lambda G$-module that is $K$-free. Denote by $\text{Ind}_m(K\lambda G)$ the set of isomorphism classes of indecomposable $K\lambda G$-modules of $K$-rank $m$. An algebra $K\lambda G$ is defined to be of SUR-type (Strongly Unbounded Representation type) if there is a function $f_\lambda : \mathbb{N} \to \mathbb{N}$ such that $f_\lambda(n) \geq n$ and $\text{Ind}_{f_\lambda(n)}(K\lambda G)$ is an infinite set for every $n > 1$. A function $f_\lambda$ is called an SUR-dimension-valued function. Given a $K\lambda H$-module $V$, we write $\text{End}_{K\lambda H}(V)$ for the ring of all $K\lambda H$-endomorphisms of $V$, $\text{rad}_{K\lambda H}(V)$ for the Jacobson radical of $\text{End}_{K\lambda H}(V)$ and $\overline{\text{End}}_{K\lambda H}(V)$ for the quotient ring

$$\overline{\text{End}}_{K\lambda H}(V) / \text{rad}_{K\lambda H}(V).$$

Given a subgroup $\Omega$ of $K^*$, we denote by $Z^2(H, \Omega)$ the group of all $\Omega$-valued normalized 2-cocycles of the group $H$, where we assume that $H$ acts trivially on $\Omega$. If $D$ is a subgroup of a group $H$, the restriction of $\lambda \in Z^2(H, K^*)$ to $D \times D$ is also denoted by $\lambda$. In this case, $K\lambda D$ is the $K$-subalgebra of $K\lambda H$ consisting of all $K$-linear combinations of the elements $\{u_d : d \in D\}$, where $\{u_h : h \in H\}$ is a canonical $K$-basis of $K\lambda H$ corresponding to $\lambda$.

Throughout the paper, $S$ denotes an arbitrary integral domain of characteristic $p$, which is not a field, $\mathfrak{m}$ is a maximal ideal of $S$ and $R = S_\mathfrak{m}$ is the localization of $S$ at $\mathfrak{m}$. The ring $R$ is a local ring and $\mathfrak{m}R$ is a unique maximal ideal of $R$. Moreover, $S/\mathfrak{m} \cong R/\mathfrak{m}R$ as fields, and as $R$-modules. Given $\mu \in Z^2(G_p, S^*)$, the kernel $\text{Ker}(\mu)$ of $\mu$ is the union of all cyclic subgroups $\langle g \rangle$ of $G_p$ such that the restriction of $\mu$ to $\langle g \rangle \times \langle g \rangle$ is a coboundary. We recall from [5, p. 196] that $G'_p \subset \text{Ker}(\mu)$, $\text{Ker}(\mu)$ is a normal subgroup of $G_p$ and the restriction of $\mu$ to $\text{Ker}(\mu) \times \text{Ker}(\mu)$ is a coboundary.

Let $H = \langle a \rangle$ be a cyclic $p$-group of order $|H| > 2$, and $K$ a commutative local ring of characteristic $p$. Suppose that there exists a non-zero element
Let $t \in \text{rad } K$ which is not a zero-divisor. Let $E_m$ be the identity matrix of order $m$, $J_m(0)$ the upper Jordan block of order $m$ with zeros on the main diagonal, and (1) the $m \times 1$-matrix of the form

$$
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}.
$$

Denote by $\Gamma_i$ the matrix $K$-representation of degree $n$ of the group $H$ defined in the following way:

1) if $n = 2$ then

$$
\Gamma_i(a) = \begin{pmatrix}
1 & t^i \\
0 & 1
\end{pmatrix} (i \in \mathbb{N});
$$

2) if $n = 3m$ ($m \geq 1$) then

$$
\Gamma_i(a) = \begin{pmatrix}
E_m & t^iE_m & J_m(0) \\
0 & E_m & t^iE_m \\
0 & 0 & E_m
\end{pmatrix} (i \in \mathbb{N});
$$

3) if $n = 3m + 1$ ($m \geq 1$) then

$$
\Gamma_i(a) = \begin{pmatrix}
E_m & t^2E_m & J_m(0) & t^1(1) \\
0 & E_m & t^iE_m & 0 \\
0 & 0 & E_m & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} (i \in \mathbb{N});
$$

4) if $n = 3m + 2$ ($m \geq 1$) then

$$
\Gamma_i(a) = \begin{pmatrix}
E_m & t^{i+2}E_m & J_m(0) & t^{2i+4}(1) & t^2(1) \\
0 & E_m & t^{2i+4}E_m & 0 & t^2(1) \\
0 & 0 & E_m & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} (i \in \mathbb{N});
$$

**Lemma 1** ([3], p. 272). Let $V_i$ be the underlying $KH$-module of the representation $\Gamma_i$. If $i \neq j$, then the $KH$-modules $V_i$ and $V_j$ are non-isomorphic. The algebra $\text{End}_{KH}(V_i)$ is finitely generated as a $K$-module and there is an algebra isomorphism $\text{End}_{KH}(V_i) \cong K / \text{rad } K$ for every $i \in \mathbb{N}$. 
Lemma 2 ([3], p. 275). Let $H = (a) \times (b)$ be an abelian group of type $(2,2)$, $t \in \text{rad } K$, $t \neq 0$ and assume that $t$ is not a zero-divisor. Denote by $V_i$ the underlying $K^H$-module of the matrix representation $\Delta_i$ of degree $n$ of the group $H$ defined as follows:

1) if $n = 2m$ ($m \geq 1$), then
$$\Delta_i(a) = \begin{pmatrix} E_m & t^i E_m \\ 0 & E_m \end{pmatrix}, \quad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) \\ 0 & E_m \end{pmatrix} \quad (i \in \mathbb{N});$$

2) if $n = 2m + 1$ ($m \geq 1$), then
$$\Delta_i(a) = \begin{pmatrix} E_m & t^i E_m & 0 \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta_i(b) = \begin{pmatrix} E_m & J_m(0) & t^i \langle 1 \rangle \\ 0 & E_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (i \in \mathbb{N});$$

If $i \neq j$, then the modules $V_i$ and $V_j$ are non-isomorphic.

Moreover, $\text{End}_{K^H}(V_i)$ is finitely generated as a $K$-module and there is an algebra isomorphism $\text{End}_{K^H}(V_i) \cong K/\text{rad } K$ for all $i \in \mathbb{N}$.

Lemma 3. The set $\sum_n(RH)$ is infinite for every integer $n > 1$.

Proof. Let $t$ be a non-zero element of $m$. Then $t \in \text{rad } R$ and $t$ is not a zero-divisor in $R$. Next apply Lemmas 1 and 2. □

Lemma 4. Let $K$ be a commutative local ring of characteristic $p$, $B$ a finite abelian $p$-group, $D$ a subgroup of $B$, $\lambda \in Z^2(B, K^\times)$ and $M$ an indecomposable $K^\lambda D$-module. Assume that $\text{End}_{K^\lambda D}(M)$ is a finitely generated $K$-algebra and $\text{End}_{K^\lambda D}(M)$ is isomorphic to a field $L$ containing $K = K/\text{rad } K$. Then
$$M^B = K^\lambda B \otimes_{K^\lambda D} M$$
is an indecomposable $K^λB$-module, $\text{End}_{K^λB}(M_B)$ is a finitely generated $K$-algebra and the quotient algebra $\text{End}_{K^λB}(M_B)$ is isomorphic to a field that is a finite purely inseparable field extension of $L$.

The proof is similar to those of Lemma 2.2 in [1].

Let $λ \in Z^2(G, S^*)$. Denote by $H_p$ the kernel of the restriction of $λ$ to $G_p \times G_p$. If $h \in H_p$ and $x \in G$, then $x^{-1}hx \in G_p$ and $|x^{-1}hx| = |h|$. From the equality $u_x^{-1}u_hu_x = γu_{x^{-1}hx}$ ($γ \in S^*$) follows

$$u_x^{-1}u_h|h|u_x = γ|h|.\ u_x^{-1}hx,$$

hence

$$u_x^{-1}hx = γ^{-|h|}u_e.$$

We obtain $x^{-1}hx \in H_p$, therefore $H_p$ is a normal subgroup of $G$. Since the restriction of $λ$ to $H_p \times H_p$ is a coboundary, we may assume that $λ_{h_1, h_2} = 1$ for all $h_1, h_2 \in H_p$. Then $γ|h| = 1$, hence $γ = 1$. Consequently, we may suppose that $λ_{a, g} = λ_{g, a} = 1$ for arbitrary $a \in H_p$ and $g \in G$.

3. ON TWISTED GROUP ALGEBRAS OF SUR-TYPE

We recall that $S$ is an integral domain of characteristic $p$, which is not a field, and $R$ is the localization of $S$ at a maximal ideal $m$. Denote by $F$ a subfield of $S$. We assume that $G$ is a finite group, and $G_p$ is a normal subgroup of $G$. Given $λ \in Z^2(G, S^*)$, we denote by $H_p$ the kernel of the restriction of $λ$ to $G_p \times G_p$.

**Theorem 1.** Let $G$ be a finite group and $λ \in Z^2(G, S^*)$. If $|H_p| > 2$ then $S^λG$ is of SUR-type with an SUR-dimension-valued function $f_λ(n) = nt_n$, where $1 \leq t_n \leq |G: H_p|$.

Proof. By Lemma 3, $\sum_n(RH_p)$ is infinite for each $n > 1$.

Let $[V] \in \sum_n(RH_p)$ and $V^G = R^λG \otimes_{RH_p} V$. Denote by $\{g_1 = e, g_2, \ldots, g_t\}$ a cross section of $H_p$ in $G$. Then

$$V^G = \bigotimes_{i=1}^t V_{i_1}, \quad V_i = u_{g_i} \otimes V.$$

The $RH_p$-module $V_i$ is called a conjugate of $V$. Denote $V^{(g_i)} = V_i$. Since $\text{End}_{RH_p}(V_i) \cong \text{End}_{RH_p}(V)$, the ring $\text{End}_{RH_p}(V_i)$ is local for every $i \in \{1, \ldots, t\}$.

Hence $V_i$ is an indecomposable $RH_p$-module. By Krull-Schmidt Theorem [[11],
Theorem 2. Let $\text{Ind}$ and $S$ is an SUR-dimension-valued function for $\text{Ind}_{p^n}(R_{H_p})$. If $V$ is isomorphic to an $R_{H_p}$-module $L^{(g)}$, then $L$ is isomorphic to the $R_{H_p}$-module $V^{(g^{-1})}$. Hence there are infinitely many classes $[L_1], \ldots, [L_i], \ldots$ in $\sum_{n}(R_{H_p})$ such that every indecomposable $R_{H_p}$-component of $(L^G)_{H_p}$ is isomorphic to none of the indecomposable $R_{H_p}$-component of $(L^G)_{H_p}$ if $i \neq j$. Therefore there are infinitely many non-isomorphic indecomposable $R^\lambda G$-modules $M$ such that $M$ is an $R^\lambda G$-component of a module of the form $V^G$. The $R$-rank of any $R^\lambda G$-component of $V^G$ is divisible by $n$ and does not exceed $n \cdot |G: H_p|$. Since

$$V \cong R \otimes_S W, \quad V^G \cong R \otimes_S W^G$$

for some $H_{p^n}$-module $W$, there exists an integer $t_n$ such that $1 \leq t_n \leq |G: H_p|$ and $\text{Ind}_{p^n}(S^\lambda G)$ is an infinite set. $\square$

Theorem 2. Let $G$ be a finite group and $\lambda \in Z^2(G, S^*\lambda)$ and assume that $|H_p: G_p| > 2$. Then $f_\lambda(n) := ndt_n$, where $d = |G_p: H_p|$ and $1 \leq t_n \leq |G: G_p|$, is an SUR-dimension-valued function for $S^\lambda G$.

Proof. Let $A = G/G'_p$ and

$$U = \bigoplus_{g \in G'_p \setminus \{e\}} S(u_g - u_e).$$

The set $V := S^\lambda G \cdot U$ is a two-sided ideal of $S^\lambda G$. The quotient algebra $S^\lambda G/V$ is isomorphic to $S^\lambda A$, where $\mu_{xG'_p, yG'_p} = \lambda_{xy}$ for all $x, y \in G$.

It contains the group algebra $SB_p$, where $B_p = H_p/G'_p$. Since $|B_p| > 2$, by Lemma 3, $\sum_{n}(RB_p)$ is infinite for each positive integer $n$. The abelian group $A_p = G_p/G'_p$ is a Sylow $p$-subgroup of $A$.

Let $[M] \in \sum_{n}(RB_p)$. By Lemma 4, the $R^\mu A_p$-module

$$M^{A_p} = R^\mu A_p \otimes_{RB_p} M$$

is indecomposable and $\text{End}_{R^\mu A_p}(M^{A_p})$ is a local ring. The $R$-rank of $M^{A_p}$ equals $n \cdot |A_p: B_p| = n \cdot |G_p: H_p|$. Arguing similarly as in the proof of Theorem 1, we conclude that if $[M]$ and $[N]$ belong to $\sum_{n}(RB_p)$ and $M \not\cong N$, then $M^{A_p} \not\cong N^{A_p}$. Let

$$(M^{A_p})^A := R^\mu A \otimes_{R^\mu A_p} M^{A_p}.$$
By the same arguments as in the proof of Theorem 1, we can prove that there exist infinitely many pairwise non-isomorphic indecomposable $R^\mu A$-modules $\Omega$ such that $\Omega$ is an $R^\mu A$-component of a module of the form $(M^\lambda p)^A$. Note that the $R$-rank of $\Omega$ is divisible by $n \cdot |G_p: H_p|$ and does not exceed

$$n \cdot |G_p: H_p| \cdot |G: G_p| = nd \cdot |G: G_p|,$$

Hence for every $n > 1$ there is an integer $t_n$ such that $1 \leq t_n \leq |G: G_p|$ and the set $\text{Ind}_{ndt_n}(S^\mu A)$ is infinite.

If $M$ is an $S^\mu A$-module, then $M$ is as well an $S^\lambda G$-module. $S^\mu A$-modules $M$ and $N$ are isomorphic if and only if $M$ and $N$ are isomorphic as $S^\lambda G$-modules. Consequently, the set $\text{Ind}_{ndt_n}(S^\lambda G)$ is also infinite for any $n > 1$. □

**Theorem 3.** Let $p \neq 2$, $G$ be a finite group and $\lambda \in Z^2(G, F^*)$. If the algebra $F^\lambda G$ is not semisimple, then the algebra $S^\lambda G$ is of SUR-type. Moreover, if $d = \dim_F(F^\lambda G_p/\text{rad} F^\lambda G_p)$ and $d < |G_p: G_p'|$, then a function $f_\lambda(n) = ndt_n$, where $1 \leq t_n \leq |G: G_p|$, is an $\text{SUR}$-dimension-valued function for $S^\lambda G$.

Proof. Applying Lemma 3 and arguing as in the proof of Theorem 2 in [3], we prove that, for every $n > 1$, there are infinitely many pairwise non-isomorphic indecomposable $R^\lambda G_p$-modules $V_1, V_2, \ldots$ satisfying the following conditions:

1) the $R$-rank of $V_1$ is equal to $nd$;
2) $\text{End}_{R^\lambda G_p}(V_i)$ is finitely generated as an $R$-module;
3) $\text{End}_{R^\lambda G_p}(V_i)$ is isomorphic to a field which is a finite purely inseparable field extension of $R/\text{rad} R$;
4) $V_i \cong R \otimes_S W_i$, where $W_i$ is an $S^\lambda G_p$-module.

Let $V_i^G := R^\lambda G \otimes_{R^\lambda G_p} V_i$ and $(V_i^G)_{G_p}$ be the module $V_i^G$ viewed as an $R^\lambda G_p$-module. The $R^\lambda G_p$-module $(V_i^G)_{G_p}$ is a direct sum of conjugates of $V_i$. By the Krull-Schmidt Theorem [[11], Sect. 7.3], $(V_i^G)_{G_p}$ has a unique decomposition into a finite sum of indecomposable $R^\lambda G_p$-modules, up to isomorphism and the order of summands. Hence the $R$-rank of each indecomposable component of $R^\lambda G$-module $V_i^G$ is divisible by $nd$. It follows that the $S$-rank of each indecomposable component of $S^\lambda G$-module $W_i^G$ is divisible by $nd$. Therefore, there exists an integer $t_n$ such that $1 \leq t_n \leq |G: G_p|$ and $\text{Ind}_{ndt_n}(S^\lambda G)$ is an infinite set. □

**Theorem 4.** Let $p = 2$, $G$ be a finite group, $\lambda \in Z^2(G, F^*)$ and moreover $d = \dim_F(F^\lambda G_2/\text{rad} F^\lambda G_2)$. 


(i) If the algebra $F^\lambda G$ is not semisimple, then the set $\text{Ind}_l(S^\lambda G)$ is infinite for some $l \leq |G|$.

(ii) If $d < \frac{1}{2}[G_2: G'_2]$, then $S^\lambda G$ is of SUR-type with an SUR-dimension-valued function $f_\lambda(n) = n dt_n$, where $1 \leq t_n \leq |G: G_2|$.

Proof. Apply Lemma 3 and proceed as in the proof of Theorem 3 in [3]. □

References


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ON LOCAL WHITNEY CONVERGENCE

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ABSTRACT

In this paper we will give definitions of local Whitney convergence in $\mathcal{F}(X,Y)$ and in $C(X,Y)$, where $X$ is a topological space, $(Y,d)$ is a metric space and $\mathcal{F}(X,Y)$ is the space of all functions from $X$ to $Y$ and $C(X,Y)$ is the space of all continuous functions from $X$ to $Y$. We will study some properties of this notion and connections with other kinds of convergence.

1. Preliminaries

Throughout the article $(X,\mathcal{T})$ will denote a $T_1$ topological space and $(Y,d)$ will denote a metric space. For any subset $A$ of the space $X$, its closure and interior will be denoted by $\operatorname{cl}(A)$ and $\operatorname{int}(A)$, respectively. Furthermore, $\mathcal{F}(X,Y)$ and $C(X,Y)$ will denote the class of functions and the class of continuous functions from $X$ to $Y$. Symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the set of real numbers, positive real numbers and positive integers, respectively. If $f: X \to Y$, $A \subset X$, then by $f|_A$ we will denote the restriction of $f$ to $A$.

Definition 1 ([1,2,4,5,6]). A sequence $(f_n)_{n \in \mathbb{N}}$ of functions from $\mathcal{F}(X,Y)$ is said to be convergent to a function $f \in \mathcal{F}(X,Y)$ in the sense of Whitney, shortly W-convergent, if for each $\varphi \in C(X,\mathbb{R}^+)$ there exists $n_0 \in \mathbb{N}$ such that $d(f_n(x),f(x)) < \varphi(x)$ for each $x \in X$ and $n \geq n_0$.

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Remark 1. It is obvious that if \((f_n)_{n \in \mathbb{N}}\) is a sequence of continuous functions which is W-convergent to \(f\) then \(f\) is continuous too.

Since each positive constant function from \(X\) to \(\mathbb{R}\) belongs to \(C(X, \mathbb{R}^+)\), W-convergence implies uniform convergence. It is well-known that if \(X\) is a pseudo-compact completely regular topological space then W-convergence and uniform convergence are equivalent. More precisely, we have the following theorem.

**Theorem 1** ([1,4]). Let \(X\) be a pseudo-compact completely regular topological space and let \(Y\) be a metrizable space. The sequence \((f_n)_{n \in \mathbb{N}}\) of functions from \(X\) to \(Y\) is W-convergent to \(f : X \to Y\) if and only if it is uniformly convergent to \(f\).

2. Local Whitney convergence

Now, we give the first definition of local Whitney convergence.

**Definition 2.** A sequence \((f_n)_{n \in \mathbb{N}}\) of functions from \(F(X, Y)\) is said to be convergent to a function \(f \in F(X, Y)\) in the sense of Whitney at a point \(x_0 \in X\), shortly W-convergent at \(x_0\), if there exists a neighborhood \(U\) of \(x_0\) such that for each \(\varphi \in C(X, \mathbb{R}^+)\) there exists \(n_0 \in \mathbb{N}\) such that

\[
d(f_n(x), f(x)) < \varphi(x)
\]

for each \(x \in U\) and \(n \geq n_0\).

We say that \((f_n)_{n \in \mathbb{N}}\) is locally W-convergent to \(f\) if it is W-convergent to \(f\) at each \(x \in X\).

**Corollary 1.** If \((f_n)_{n \in \mathbb{N}}\) is a sequence of continuous functions which is locally W-convergent to \(f\) then \(f\) is continuous function too.

It follows directly from the definitions, that W-convergence implies local W-convergence. Later, we will give an example of locally W-convergent sequence of continuous functions which is not W-convergent.

**Theorem 2.** Let \(f_n \in F(X, Y)\) for \(n \in \mathbb{N}\) and let \(f \in F(X, Y)\). If the sequence \((f_n)_{n \in \mathbb{N}}\) is locally W-convergent to \(f\) then \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\).
Proof. Let \((f_n)_{n \in \mathbb{N}}\) be locally W-convergent to \(f\). Take any \(\varepsilon > 0\) and any compact subset \(A\) of \(X\). Let \(\varphi \in C(X, \mathbb{R}^+)\) be a constant function, \(\varphi(x) = \varepsilon\) for \(x \in X\).

Then for each \(x \in A\) we can find a neighborhood \(U_x\) of \(x\) and \(n_x \in \mathbb{N}\) such that
\[d(f_n(y), f(y)) < \varphi(y) = \varepsilon\]
for \(y \in U_x\) and \(n \geq n_x\).

Since \(A\) is compact, \(A \subset U_{x_1} \cup \ldots \cup U_{x_k}\) for some \(x_1, \ldots, x_k \in A\).

Let \(n_0 = \max\{n_{x_1}, \ldots, n_{x_k}\}\). Then \(d(f_n(x), f(x)) < \varepsilon\) for \(x \in A\) and \(n \geq n_0\). This completes the proof. \(\square\)

**Theorem 3.** Let \((X, T)\) be a paracompact space. The following conditions are equivalent:

1. For each metric space \((Z, \varrho)\), every sequence \((f_n)_{n \in \mathbb{N}}\) from \(F(X, Z)\) is locally W-convergent to \(f \in F(X, Z)\) if and only if \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\),
2. Every sequence \((f_n)_{n \in \mathbb{N}}\) of functions from \(F(X, \mathbb{R})\) is locally W-convergent to \(f \in F(X, \mathbb{R})\) if and only if \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\),
3. For each metric space \((Z, \varrho)\) every sequence \((f_n)_{n \in \mathbb{N}}\) from \(C(X, Z)\) is locally W-convergent to \(f \in C(X, Z)\) if and only if \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\),
4. Every sequence \((f_n)_{n \in \mathbb{N}}\) from \(C(X, \mathbb{R})\) is locally W-convergent to \(f\) from \(C(X, \mathbb{R})\) if and only if \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\),
5. \(X\) is locally compact.

**Proof.** The implications \((1) \Rightarrow (2), (3) \Rightarrow (4), (1) \Rightarrow (3)\) and \((2) \Rightarrow (4)\) are obvious.

\((4) \Rightarrow (5)\) Let local W-convergence and almost uniform convergence in \(C(X, \mathbb{R})\) be equivalent. Suppose that \(X\) is not locally compact at some \(x_0 \in X\). Let \(f_n(x) = \frac{1}{n^2}, f(x) = 0\) for \(n \in \mathbb{N}, x \in X\).

Obviously, the sequence \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\) (actually, \((f_n)_{n \in \mathbb{N}}\) is uniformly convergent to \(f\)). Let \(U\) be any neighborhood of \(x_0\). Then \(\text{cl}(U)\) is not compact and since \(X\) is paracompact, it is not pseudo-compact.
Thus there exists \( \tilde{\varphi} \in C(\text{cl}(U), \mathbb{R}^+) \) such that \( \inf_{x \in \text{cl}(U)} \tilde{\varphi}(x) = 0 \). Hence \( \inf_{x \in U} \tilde{\varphi}(x) = 0 \). By normality of \( X \) (\( X \) is normal, since it is paracompact), there exists \( \varphi \in C(X, \mathbb{R}^+) \) such that \( \varphi|_{\text{cl}(U)} = \tilde{\varphi} \). Then for each \( n \in \mathbb{N} \) we have \( |f_n(x) - f(x)| = \frac{1}{n} \geq \varphi(x) \) for some \( x \in U \).

Hence the sequence \( (f_n)_{n \in \mathbb{N}} \) is not locally \( W \)-convergent to \( f \). This contradicts to assumptions. Thus \( X \) is locally compact.

(5) \( \Rightarrow \) (1) Let \( X \) be locally compact and let \((Z, g)\) be any metric space. Take any sequence \( (f_n)_{n \in \mathbb{N}} \) from \( \mathcal{F}(X, Z) \) and \( f \in \mathcal{F}(X, Z) \). If the sequence is locally \( W \)-convergent to \( f \) then, by Theorem 2, it is almost uniformly convergent to \( f \). Assume that \( (f_n)_{n \in \mathbb{N}} \) is almost uniformly convergent to \( f \). Let \( x_0 \in X \). By local compactness of \( X \) there exists a neighborhood \( U \) of \( x_0 \) such that \( \text{cl}(U) \) is compact. Then \( (f_n|_{\text{cl}(U)})_{n \in \mathbb{N}} \) is uniformly convergent to \( f|_{\text{cl}(U)} \). Let \( \varphi \in C(X, \mathbb{R}^+) \). Then \( \inf_{x \in \text{cl}(U)} \varphi(x) = c > 0 \) and there exists \( n_0 \in \mathbb{N} \) such that \( g(f_n(x), f(x)) < c \leq \varphi(x) \) for \( x \in \text{cl}(U) \) and \( n \geq n_0 \). It follows that the sequence \( (f_n)_{n \in \mathbb{N}} \) is locally \( W \)-convergent to \( f \). The proof is completed. \( \square \)

**Theorem 4.** Let \( (X, T) \) be a normal space and let \( f_n \in \mathcal{F}(X, Y) \) for \( n \in \mathbb{N} \) and \( f \in \mathcal{F}(X, Y) \). For each \( x_0 \in X \) the following conditions are equivalent:

(1) the sequence \( (f_n)_{n \in \mathbb{N}} \) is \( W \)-convergent to \( f \) at \( x_0 \),

(2) there exists a neighborhood \( U \) of \( x_0 \) such that the sequence \( (f_n|_{\text{cl}(U)})_{n \in \mathbb{N}} \) is \( W \)-convergent to \( f|_{\text{cl}(U)} \),

(3) for each neighborhood \( U \) of \( x_0 \) there exists a neighborhood \( V \subset U \) of \( x_0 \) such that \( (f_n|_{\text{cl}(V)})_{n \in \mathbb{N}} \) is \( W \)-convergent to \( f|_{\text{cl}(V)} \),

(4) there exists a neighborhood \( U \) of \( x_0 \) such that for each neighborhood \( V \subset U \) of \( x \) the sequence \( (f_n|_{\text{cl}(V)})_{n \in \mathbb{N}} \) is \( W \)-convergent to \( f|_{\text{cl}(V)} \). 

**Proof.**

2) \( \Rightarrow \) 1) Let \( U \) be a neighborhood of a point \( x_0 \) such that the sequence \( (f_n|_{\text{cl}(U)})_{n \in \mathbb{N}} \) is \( W \)-convergent to \( f|_{\text{cl}(U)} \). Take any \( \varphi \in C(X, \mathbb{R}^+) \). Then \( \varphi|_{\text{cl}(U)} \in C(\text{cl}(U), \mathbb{R}^+) \). Therefore there exists \( n_0 \in \mathbb{N} \) such that

\[
d(f_n(x), f(x)) < \varphi|_{\text{cl}(U)}(x)
\]

for each \( x \in \text{cl}(U) \) and \( n \geq n_0 \). In particular,

\[
d(f_n(x), f(x)) < \varphi(x)
\]
for each $x \in U$ and $n \geq n_0$. It follows that $(f_n)_{n \in \mathbb{N}}$ is $W$-convergent to $f$ at $x_0$.

3) $\Rightarrow$ 2) This implication is evident.

4) $\Rightarrow$ 3) This implication is evident too.

1) $\Rightarrow$ 4) Let the sequence $(f_n)_{n \in \mathbb{N}}$ be $W$-convergent to $f$ at $x_0$. There exists a neighborhood $U$ of $x_0$ such that for $\varphi \in C(X, \mathbb{R}^+)$ we can find $n_0 \in \mathbb{N}$ for which $d(f_n(x), f(x)) < \varphi(x)$ for $n \geq n_0$ and $x \in U$. Since $X$ is normal, there exists a neighborhood $U_1$ of $x_0$ such that $\text{cl}(U_1) \subset U$. Let $V$ be any neighborhood of $x_0$ contained in $U_1$. Then $\text{cl}(V) \subset \text{cl}(U_1) \subset U$.

We claim that the sequence $(f_n|_{\text{cl}(V)})_{n \in \mathbb{N}}$ is $W$-convergent to $f|_{\text{cl}(V)}$. Let $\tilde{\varphi} \in C(\text{cl}(V), \mathbb{R}^+)$. Since $X$ is normal, there exists $\varphi \in C(X, \mathbb{R}^+)$ such that $\varphi|_{\text{cl}(V)} = \tilde{\varphi}$. Then we can find $n_0 \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varphi(x) = \tilde{\varphi}(x)$$

for $x \in U$ and $n \geq n_0$. In particular, $d(f_n(x), f(x)) < \varphi(x) = \tilde{\varphi}(x)$ for $x \in \text{cl}(V)$ and $n \geq n_0$. Thus $(f_n|_{\text{cl}(V)})_{n \in \mathbb{N}}$ is $W$-convergent to $f|_{\text{cl}(V)}$ and the proof is completed. □

We will need the following result.

**Theorem 5** ([3, Theorem 4]). Let $(X, \tau)$ be a normal topological space, $(Y, d)$ a metric space. Let $f \in C(X, Y)$ and $f_n \in C(X, Y)$ for $n \in \mathbb{N}$. Then the sequence $(f_n)_{n \in \mathbb{N}}$ is $W$-convergent to $f$ if and only if the following two conditions hold:

1. the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent to $f$;
2. there exists a closed countably compact set $K \subset X$ such that if $U$ is an open subset of $X$ containing $K$ then we can find $n_0 \in \mathbb{N}$ such that $f_n|_{(X \setminus U)} = f|_{(X \setminus U)}$ for $n \geq n_0$.

**Problem 1.** Is Theorem 5 true for functions from $\mathcal{F}(X, Y)$?

**Theorem 6.** Let $X$ be a paracompact space. Assume that $f_n \in C(X, Y)$ for $n \in \mathbb{N}$ and $f \in C(X, Y)$. Then the sequence $(f_n)_{n \in \mathbb{N}}$ is locally $W$-convergent to $f$ if and only if it is almost uniformly convergent and there exist a locally compact set $F \subset X$ and a sequence $(A_n)_{n \in \mathbb{N}}$ of open subsets of $X$ such that

1. $f_k(x) = f(x)$ for all $x \in A_n$ and $k \geq n$, 

Proof. First assume that \((f_n)_{n \in \mathbb{N}}\) is locally W-convergent to \(f\). Obviously, \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent. By Theorem 4, for each \(x \in X\) we can find a neighborhood \(\tilde{U}_x\) of \(x\) such that \(\left( f_n|_{\cl(\tilde{U}_x)} \right)_{n \in \mathbb{N}}\) is Whitney convergent to \(f|_{\cl(\tilde{U}_x)}\). By Theorem 4 and by normality of \(X\), there exists a neighborhood \(U_x\) of \(x\) such that \(\cl(U_x) \subset \tilde{U}_x\) and \(\left( f_n|_{\cl(U_x)} \right)_{n \in \mathbb{N}}\) is Whitney convergent to \(f|_{\cl(U_x)}\).

Let \(F_x \subset \cl(\tilde{U}_x)\) be a closed countably compact set which satisfies conditions of Theorem 5 for the sequence \(\left( f_n|_{\cl(\tilde{U}_x)} \right)_{n \in \mathbb{N}}\). By paracompactness of \(X\), \(F_x\) is locally compact.

Take any \(y \in \cl(U_x) \setminus F_x \subset \tilde{U}_x \setminus F_x\). By normality of \(X\), there exist a neighborhood \(V\) of \(y\) such that \(\cl(V) \cap F_x = \emptyset\) and \(\cl(V) \subset \tilde{U}_x\). Then, by Theorem 5, we can find \(n_0 \in \mathbb{N}\) such that \(f_n(t) = f(t)\) for each \(t \in V\) and each \(n \geq n_0\). Thus we have proven that

\((*)\) for each \(y \in \cl(U_x) \setminus F_x\) there exist a neighborhood \(V\) of \(y\) and \(n_0 \in \mathbb{N}\) such that \(f_n(t) = f(t)\) for each \(t \in V\) and each \(n \geq n_0\).

Put

\[ A_n = \text{int}\{ x \in X : f_k(x) = f(x) \text{ for each } k \geq n \}\]

for \(n \in \mathbb{N}\) and

\[ F = X \setminus \bigcup_{n=1}^{\infty} A_n.\]

Then \(F\) is closed and, by \((*)\), \(F \cap \cl(U_x) \subset F_x\) for \(x \in X\). Hence \(F\) is locally compact. Take any \(x \in F\) and put \(G_x = U_x\). Then condition (3) is satisfied, by Theorem 5 and by definition of \(U_x\).

Now, assume that \((f_n)_{n \in \mathbb{N}}\) is almost uniformly convergent to \(f\) and there exist a locally compact set \(F \subset X\) and a sequence \((A_n)_{n \in \mathbb{N}}\) of open subsets of \(X\) for which conditions (1), (2) and (3) hold. Take any \(x_0 \in X\). First consider the case, where \(x_0 \notin F\).

Then \(x_0 \in A_k\) for some \(k \in \mathbb{N}\). Since \(X\) is normal, there exists a neighborhood \(U\) of \(x_0\) such that \(\cl(U) \subset A_k\). Then \(f_n|_{\cl(U)} = f|_{\cl(U)}\) for \(n \geq k\). Hence
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\((f_n|\text{cl}(U))_{n \in \mathbb{N}}\) is Whitney convergent to \(f|\text{cl}(U)\). By Theorem 4, \((f_n)_{n \in \mathbb{N}}\) is W-convergent to \(f\) at \(x_0\).

Finally, consider the case, where \(x_0 \in F\). Let \(G_{x_0}\) be a neighborhood of \(x_0\) such that for any neighborhood \(V\) of \(F \cap G_{x_0}\), there exists \(n_0\) such that \(f_n(x) = f(x)\) for all \(x \in G_{x_0} \setminus V\) and \(n \geq n_0\).

Since \(F\) is locally compact and \(X\) is normal, there exists a neighborhood \(W_{x_0}\) of \(x_0\) such that \(\text{cl}(W_{x_0}) \cap F\) is compact and \(\text{cl}(W_{x_0}) \subset G_{x_0}\). Consider the sequence \((f_n|\text{cl}(W_{x_0}))_{n \in \mathbb{N}}\). Clearly, it is uniformly convergent to \(f|\text{cl}(W_{x_0})\) and \(F \cap \text{cl}(W_{x_0})\) is compact. Let \(V\) be any neighborhood of \(F \cap \text{cl}(W_{x_0})\) in \(\text{cl}(W_{x_0})\). There exists open subset \(\tilde{V}\) of \(X\) such that \(V = \tilde{V} \cap \text{cl}(W_{x_0})\).

Then \((G_{x_0} \setminus \text{cl}(W_{x_0})) \cup \tilde{V})\) is a neighborhood of \(F \cap G_{x_0}\). Therefore we can find \(n_0\) such that \(f_n|\text{cl}(W_{x_0})(t) = f|\text{cl}(W_{x_0})(t)\) for each \(t \in \text{cl}(W_{x_0}) \setminus V\). By Theorem 5, \((f_n|\text{cl}(W_{x_0}))_{n \in \mathbb{N}}\) is Whitney convergent to \(f|\text{cl}(W_{x_0})\) and, by Theorem 4, \((f_n)_{n \in \mathbb{N}}\) is W-convergent to \(f\) at \(x_0\).

Since \(x_0\) was arbitrary, the proof is completed. □

3. STRONG LOCAL W-CONVERGENCE AND W*-CONVERGENCE

Theorem 4 motivates us to introduce two new definitions of local Whitney convergence.

**Definition 3.** A sequence \((f_n)_{n \in \mathbb{N}}\) of functions from \(F(X, Y)\) is said to be w*-convergent to \(f \in F(X, Y)\) at a point \(x_0 \in X\), if there exists a neighborhood \(U\) of \(x_0\) such that for each \(\varphi \in C(U, \mathbb{R}^+)\) there exists \(n_0 \in \mathbb{N}\) such that

\[d(f_n(x), f(x)) < \varphi(x)\]

for each \(x \in U\) and \(n \geq n_0\).

Equivalently, \((f_n)_{n \in \mathbb{N}}\) is w*-convergent to a function \(f \in F(X, Y)\) at a point \(x_0 \in X\), if there exists a neighborhood \(U\) of the point \(x_0\) such that \((f_n|U)_{n \in \mathbb{N}}\) is W-convergent to \(f|U\).

We say that \((f_n)_{n \in \mathbb{N}}\) is locally w*-convergent to \(f\) if it is w*-convergent to \(f\) at every \(x \in X\).

**Definition 4.** A sequence \((f_n)_{n \in \mathbb{N}}\) of functions from \(F(X, Y)\) is said to be W-convergent in the strong sense to \(f \in F(X, Y)\) at a point \(x_0 \in X\), shortly SW-convergent at \(x_0\), if for each neighborhood \(U\) of \(x_0\) we can find
a neighborhood $V \subset U$ of $x_0$ such that for each $\varphi \in C(V, \mathbb{R}^+)$ there exists $n_0 \in \mathbb{N}$ such that
\[
d(f_n(x), f(x)) < \varphi(x)
\]
for all $x \in V$ and $n \geq n_0$.

Equivalently, $(f_n)_{n \in \mathbb{N}}$ is SW-convergent to a function $f \in F(X, Y)$ at a point $x_0 \in X$, if for each neighborhood $U$ of $x_0$ there exists a neighborhood $V \subset U$ of $x_0$ such that the sequence $(f_n|_V)_{n \in \mathbb{N}}$ is W-convergent to $f|_V$.

We say that $(f_n)_{n \in \mathbb{N}}$ is locally SW-convergent to $f$ if it is SW-convergent to $f$ at every $x \in X$.

**Corollary 2.** If a sequence $(f_n)_{n \in \mathbb{N}}$ of continuous functions from $X$ to $Y$ is locally SW-convergent or locally $w^*$-convergent to $f$ then $f : X \to Y$ is continuous too.

From the definitions we easily get the following two propositions.

**Proposition 1.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence from $F(X, Y)$, $f \in F(X, Y)$ and $x_0 \in X$.

(1) If $(f_n)_{n \in \mathbb{N}}$ is $w^*$-convergent to $f$ at $x_0$ then $(f_n)_{n \in \mathbb{N}}$ is W-convergent to $f$ at $x_0$.

(2) If $(f_n)_{n \in \mathbb{N}}$ is SW-convergent to $f$ at $x_0$ then $(f_n)_{n \in \mathbb{N}}$ is $w^*$-convergent to $f$ at $x_0$.

The first relation follows from the obvious fact that if $\varphi \in C(X, \mathbb{R}^+)$ then $\varphi|_U \in C(U, \mathbb{R}^+)$. 

**Proposition 2.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from $F(X, Y)$ and $f \in F(X, Y)$. If $(f_n)_{n \in \mathbb{N}}$ is W-convergent to function $f$ then $(f_n)_{n \in \mathbb{N}}$ is locally $w^*$-convergent to $f$.

We can put $U = X$ in the definition of $w^*$-convergence.

Relationships between discussed types of convergence can be illustrated in the following diagrams.

\[
\text{SW-conv. at } x_0 \Rightarrow w^*\text{-conv. at } x_0 \Rightarrow \text{W-conv. at } x_0
\]
W-convergence

\[\downarrow \]

local SW-conv. \Rightarrow local \ w^*\text{-conv.} \Rightarrow local W-conv.

We will show that none of the reverse implications hold, even in \(C(X,Y)\).

**Example 1.** Let \(f_n : \mathbb{R} \to \mathbb{R}\),

\[
f_n(x) = \begin{cases} 
0 & \text{if } x \leq n, \\
\frac{1}{n} & \text{if } x \geq n + 1, \\
\frac{x-n}{n} & \text{if } n < x < n + 1, 
\end{cases}
\]

for \(n \in \mathbb{N}\) and let \(f : \mathbb{R} \to \mathbb{R}\), \(f(x) = 0\) if \(x \in \mathbb{R}\).

Then \(f\) and \(f_n\) for \(n \in \mathbb{N}\) are continuous. Since for each \(x_0 \in \mathbb{R}\) there exist \(n_0 \in \mathbb{N}\) such that \(f_n(x) = 0\) if \(n \geq n_0\) and \(x \in (x_0 - 1, x_0 + 1)\), the sequence \((f_n)_{n \in \mathbb{N}}\) is locally SW-convergent to \(f\). On the other hand, there exists \(\varphi \in C(\mathbb{R}, \mathbb{R}^+)\) such that \(\varphi(n + 1) = \frac{1}{n}\) for \(n \geq 1\). Then \(|f_n(n + 1) - f(n + 1)| = \frac{1}{n} = \varphi(n + 1)\) for each \(n\). It follows that \((f_n)_{n \in \mathbb{N}}\) is not W-convergent to \(f\).

**Example 2.** Let \(f_n : [0,1] \to \mathbb{R}\), \(f_n(x) = \frac{1}{n}\) if \(x \in [0,1]\), \(n \in \mathbb{N}\) and let \(f : [0,1] \to \mathbb{R}\), \(f(x) = 0\) if \(x \in [0,1]\). Then functions \(f\) and \(f_n\) are continuous. Since \([0,1]\) is compact and \((f_n)_{n \in \mathbb{N}}\) is uniformly convergent to \(f\), \((f_n)_{n \in \mathbb{N}}\) is W-convergent to \(f\).

Take any \(x \in (0,1)\) and let \(U = (0,1)\). Then \(U\) is a neighborhood of \(x\) and any neighborhood \(V\) of \(x\) contained in \(U\) is not closed. Therefore there exist \(\varphi \in C(V, \mathbb{R}^+)\) and a sequence \((x_n)_{n \in \mathbb{N}} \subset V\) such that \(\varphi(x_n) = \frac{1}{n}\). Hence \((f_n|_V)_{n \in \mathbb{N}}\) is not W-convergent to \(f|_V\). Thus \((f_n)_{n \in \mathbb{N}}\) is not SW-convergent to \(f\) at any \(x \in (0,1)\). Hence \((f_n)_{n \in \mathbb{N}}\) is not locally SW-convergent to \(f\).

**Example 3.** Let \(f_n : \mathbb{R} \to \mathbb{R}\), \(f_n(x) = \frac{1}{n}\) if \(n \in \mathbb{N}\), \(x \in \mathbb{R}\) and let \(f : \mathbb{R} \to \mathbb{R}\), \(f(x) = 0\) if \(x \in \mathbb{R}\). Obviously, \((f_n)_{n \in \mathbb{N}}\) is not \(w^*\)-convergent to \(f\) at any point \(x \in \mathbb{R}\) (arguments are similar as in the previous example).

Let \(x_0 \in \mathbb{R}\) and \(\varphi \in C(\mathbb{R}, \mathbb{R}^+)\). Then there exists \(\delta > 0\) such that \(\varphi(x) \geq \delta\) for \(x \in (x_0 - 1, x_0 + 1)\). Hence \(|f_n(x) - f(x)| = \frac{1}{n} < \delta \leq \varphi(x)\) for \(x \in (x_0 - 1, x_0 + 1)\) and sufficiently large \(n\). Thus \((f_n)_{n \in \mathbb{N}}\) is locally W-convergent.
The next example shows that \( w^* \)-convergence at a point is not a local property.

**Example 4.** Let \( f_n, g_n : [0, 2] \to \mathbb{R} \), \( f_n(x) = \frac{1}{n} \) if \( x \in [0, 2] \) and

\[
g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \{0\} \cup \left[ \frac{1}{n}, 2 \right], \\ 1 & \text{if } x = \frac{1}{2n}, \\ \text{linear in } [0, \frac{1}{2n}] \text{ and } \left[ \frac{1}{2n}, \frac{1}{n} \right], \end{cases}
\]

if \( n \in \mathbb{N} \). Next, let \( f : [0, 2] \to \mathbb{R} \), \( f(x) = 0 \) if \( x \in [0, 2] \). Since \([0, 2]\) is compact and \((f_n)_{n \in \mathbb{N}}\) is uniformly convergent to \( f \), \((f_n)_{n \in \mathbb{N}}\) is locally \( w^* \)-convergent to \( f \).

Take any \( x \in [0, 2] \). Let \( U \) be any neighborhood of \( x \). Then either \([0, \eta) \subset U\) for some \( \eta > 0 \) or \( U \) is not closed subset of \([0, 2]\).

If \([0, \eta) \subset U\) for some \( \eta > 0 \) then \((g_n|U)_{n \in \mathbb{N}}\) is not uniformly convergent. Hence \((g_n|U)_{n \in \mathbb{N}}\) is not Whitney convergent.

Finally, assume that \( U \) is not a closed subset of \([0, 2]\). Then there exist a sequence \((x_n)_{n \in \mathbb{N}} \subset U\) and \( \varphi \in C(U, \mathbb{R}^+) \) such that \( \varphi(x_n) = \frac{1}{n} \) for \( n \geq 1 \).

Since \( g(x) \geq \frac{1}{n} \) for \( x \in [0, 2] \), we have

\[
|g_n(x_n) - f(x_n)| \geq \frac{1}{n} = \varphi(x_n).
\]

It follows that \((g_n|U)_{n \in \mathbb{N}}\) is not Whitney convergent. We have proven that \((g_n)_{n \in \mathbb{N}}\) is not \( w^* \)-convergent at any \( x \in [0, 2] \). Thus the sequence \((f_n)_{n \in \mathbb{N}}\) is \( w^* \)-convergent at each point \( x \in [0, 2] \) and \((g_n)_{n \in \mathbb{N}}\) is not \( w^* \)-convergent at any \( x \in [0, 2] \), nevertheless \( f_n(x) = g_n(x) \) for all \( x \in [1, 2] \) and \( n \in \mathbb{N} \).

**Theorem 7.** Let \((X, T)\) be a paracompact perfect topological space, \( x_0 \in X \) and let \((Y, d)\) be any metric space. If \( X \) is locally compact at \( x_0 \) then the following conditions are equivalent:

1. \( W \)-convergence at \( x_0 \) and \( SW \)-convergence at \( x_0 \) in \( \mathcal{F}(X, Y) \) are equivalent.
2. \( W \)-convergence at \( x_0 \) and \( SW \)-convergence at \( x_0 \) in \( \mathcal{C}(X, Y) \) are equivalent.
3. there exists a local base of \( T \) at \( x_0 \) consisting of sets which are both open and closed.
Proof. 3) $\Rightarrow$ 1) Assume that there exists a local base at $x_0$ consisting of sets which are both open and closed. Let $(f_n)_{n \in \mathbb{N}}$ be any sequence from $\mathcal{F}(X,Y)$ and $f \in \mathcal{F}(X,Y)$. If $(f_n)_{n \in \mathbb{N}}$ is SW-convergent to $f$ at $x_0$ then, obviously, it is $W$-convergent at $x_0$.

On the other hand, assume that $(f_n)_{n \in \mathbb{N}}$ is $W$-convergent to $f$ at $x_0$ and let $U$ be any neighborhood of $x_0$. By Theorem 4, there exists a neighborhood $U_1$ of $x_0$ such that for each neighborhood $U_2$ of $x_0$ contained in $U_1$ the sequence $(f_n|_{U_2})_{n \in \mathbb{N}}$ is $W$-convergent to $f|_{U_2}$. By assumption, we can find a neighborhood $V$ of $x_0$ such that $V \subseteq U_1 \cap U$ and $\text{cl}(V) = V$. Then $(f_n|_V)_{n \in \mathbb{N}}$ is $W$-convergent to $f|_V$. It follows that $(f_n)_{n \in \mathbb{N}}$ is SW-convergent to $f$ at $x_0$. Thus $W$-convergence at $x_0$ and SW-convergence at $x_0$ in $\mathcal{F}(X,Y)$ are equivalent.

1) $\Rightarrow$ 2) This implication is obvious.

2) $\Rightarrow$ 3) Assume that $W$-convergence at $x_0$ and SW-convergence at $x_0$ are equivalent in $C(X,Y)$. Suppose that there exists a neighborhood $U$ of $x_0$ such that each neighborhood $V$ of $x_0$ contained in $U$ is not closed. By local compactness of $X$ at $x_0$, there exists a neighborhood $U_0$ of $x_0$ with compact closure. Let $f : X \to Y$, $f(x) = 0$ for $x \in X$. By normality of $X$, we can find a neighborhood $G$ of $x_0$ and a sequence of continuous functions $f_n : X \to Y$ such that $\text{cl}(G) \subset U \cap U_0$, $f_n(x) = \frac{1}{n}$ for $x \in G$, $f_n(x) = 0$ for $x \in X \setminus (U \cap U_0)$ and $0 \leq f_n(x) \leq \frac{1}{n}$ for $x \in X$. By Theorem 5, $(f_n)_{n \in \mathbb{N}}$ is Whitney convergent to $f$, because $\text{cl}(U \cap U_0)$ is compact. Hence $(f_n)_{n \in \mathbb{N}}$ is $W$-convergent at $x_0$.

Let $V \subseteq G$ be any neighborhood of $x_0$. In particular, $V \subseteq U$. Therefore $V$ is not closed. It follows that $V$ is not compact. Since $X$ is perfect, $V$ is of type $F_{\sigma}$. Therefore $V$ is a paracompact space. It follows that $V$ is non pseudo-compact. Hence there exist a sequence $(x_n)_{n \in \mathbb{N}}$ of points from $V$ and $\varphi \in C(V, \mathbb{R}^+)$ such that $\varphi(x_n) = \frac{1}{n}$ for $n \geq 1$. Then

$$d(f_n(x_n), f(x_n)) = \frac{1}{n} = \varphi(x_n)$$

for all $n$. It follows that $(f_n|_V)_{n \in \mathbb{N}}$ is not Whitney convergent to $f|_V$ and therefore $(f_n)_{n \in \mathbb{N}}$ is not SW-convergent to $f$ at $x_0$, which is a contradiction. This completes the proof. $\square$
Since every metric space is paracompact and perfect we have the following corollary.

**Corollary 3.** Let \((X, d_X)\) be a locally compact metric space. The following conditions are equivalent:

1. for each metric space \((Z, \varrho)\) local W-convergence and local SW-convergence in \(\mathcal{F}(X, Z)\) are equivalent.
2. local W-convergence and local SW-convergence in \(\mathcal{F}(X, \mathbb{R})\) are equivalent.
3. for each metric space \((Z, \varrho)\) local W-convergence and local SW-convergence in \(C(X, Z)\) are equivalent.
4. local W-convergence and local SW-convergence in \(C(X, \mathbb{R})\) are equivalent.
5. \(X\) is zero dimensional.

**References**


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MODULAR TECHNIQUE OF HIGH-SPEED PARALLEL COMPUTING ON THE SETS OF POLYNOMIALS

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ABSTRACT

In this paper we present the modular computing structures (MCS) defined on the set of polynomials over finite rings of integers. This article is a continuation of research on the development of modular number systems (MNS) on arbitrary mathematical structures such as finite groups, rings and Galois fields [1-7].

1. INTRODUCTION

At the present time in the modern computer algebra, digital signal processing, coding theory, cryptography, many others fields of science and engineering the polynomial operations are of great importance. Therefore, studies on the development of modular technique of information processing in the direction of optimization the parallel computing structures defined on the polynomial ranges are of the utmost significance.

The developed technique of minimal redundant modular codification of ranges with vectorial structure is based on the introduction of minimal redundancy at a lower level (a level of real components) [1-3]. This universal and effective basis for synthesis of computer arithmetic procedures for the algebraic systems with polynomial carriers.

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The technique of interval-modular forms used for the real components of elements of coded ranges as well as the calculated relations for the interval-index characteristics of integer real numbers are the key elements of the proposed methodology [2, 3]. This allows us to create on the basis of the real minimal redundant modular systems the required variants of computer arithmetic for polynomial modular number systems under consideration.

2. SOME THEORETICAL FOUNDATIONS

Let us consider the set \( \mathbb{Z}[x] \) of all polynomials of finite degree with coefficients in the ring \( \mathbb{Z} \) of integers and the variable \( x \). This set is a commutative ring with unity \( e(x) = 1 \) and zero \( 0(x) \).

**Definition 1.** If the set of divisors of some element \( f(x) \) of natural degree from the ring \( \mathbb{Z}[x] \) is confined to polynomials of the form \( Cd(x) \) such that \( f(x) = Cd(x) \), where \( C \in \mathbb{Z} \), then the polynomial \( f(x) \) is called irreducible.

**Definition 2.** The common divisor \( d(x) \) of polynomials \( p_1(x), p_2(x), \ldots, p_n(x) \), \( (n \geq 1) \), divisible by any other of their common divisor is called the least common divisor of polynomials and is denoted by

\[
d(x) = \langle p_1(x), p_2(x), \ldots, p_n(x) \rangle.
\]

In the case \( d(x) = 1 \), the polynomials \( p_1(x), p_2(x), \ldots, p_n(x) \) are called pairwise coprime.

**Definition 3.** The polynomial from \( \mathbb{Z}[x] \) with unitary coefficient at the high-order degree of \( x \) is called the normalized polynomial.

Following the offered technique of constructing a MNS [3] in this case first of all requires the creation of the complete set of residues (CSR) for the factor ring \( \mathbb{Z}[x]/(p_l(x)) \) \( (l = 1, 2, \ldots, n; \ n \geq 2) \) with respect to selected pairwise relatively prime polynomial modules \( p_1(x), p_2(x), \ldots, p_n(x) \) generating in \( \mathbb{Z}[x] \) principal ideals \( (p_1(x)), (p_2(x)), \ldots, (p_n(x)) \). At the same time the governing equivalence relation is actually given by Euclidean lemma [1, 9] formulated as follows.

**Lemma 1.** For any polynomial \( f(x) \) in \( \mathbb{Z}[x] \) and polynomial modules \( p(x) \) when \( \deg p(x) \geq 1 \) there are unique elements \( q(x) \) and \( r(x) \) such that

\[
f(x) = q(x) \cdot p(x) + r(x) \quad (\deg r(x) < \deg p(x)).
\]
3. Polynomial modular number system

As in computer applications the finite mathematical structures are used, then for the construction of polynomial MNS (PMNS) instead of the ring \( \mathbb{Z}[x] \) the ranges of the form

\[
\mathbb{Z}^s_m[x] = \left\{ A(x) = \sum_{j=0}^s a_j x^j \mid (a_0, a_1, \ldots, a_{s-1}) \in (\mathbb{Z}_m \times \mathbb{Z}_m \times \cdots \times \mathbb{Z}_m), \right\},
\]

are used, where \( m \) and \( s \) are the fixed positive integers; \( m \geq 2 \). The cardinality of the set \( \mathbb{Z}^s_m[x] \) is equal to \( N = |\mathbb{Z}^s_m[x]| = m^s \).

Let \( \mathbb{Z}_m \) be the set of all polynomials over the ring \( \mathbb{Z} \), and \( p(x) \) be any element of \( s \)th degree from \( \mathbb{Z}_m \). Then according to Euclidean lemma (which is valid also for the ring \( \mathbb{Z}_m \)) the set \( \mathbb{Z}^s_m[x] \) coincides with the set of all residual \( r(x) \) of division of \( f(x) \) by \( p(x) \) (see (1)), while \( f(x) \) represents every element from the set \( \mathbb{Z}_m \). Thus, the ring \( \mathbb{Z}^s_m[x] \) is a CSR modulo \( p(x) \). For the CSR of this type a notation \( \langle \cdot \rangle_{p(x)} \) is used. The operation modulo \( p(x) \) over the polynomial \( f(x) \), is designated as \( \langle f(x) \rangle_{p(x)} \). It is also quite clear that any two rings \( \langle \cdot \rangle_{p(x)} \) and \( \langle \cdot \rangle_{g(x)} \) modulo \( p(x) \) and \( g(x) \) of the same degree \( \langle p(x), g(x) \in \mathbb{Z}^s_m[x] \rangle, \deg p(x) = \deg g(x) \), respectively, are automorphic (i.e. are isomorphic and have the same carrier).

On this basis, in the general case the PMNS with pairwise relatively prime polynomial modules \( (p_1(x), p_2(x), \ldots, p_n(x)) \) is defined as an algebraic system

\[
S_{PMNS} = \langle \mathbb{Z}_m, \langle \cdot \rangle_{p_1(x)}, \langle \cdot \rangle_{p_2(x)}, \ldots, \langle \cdot \rangle_{p_n(x)}; (+, +, \ldots, +), (\cdot, \cdot, \ldots, \cdot) \rangle,
\]

(2)

where \( P(x) = \prod_{l=1}^n p_l(x) \).

The isomorphism \( \phi : P(x) \to p_1(x) \times p_2(x) \times \cdots \times p_n(x) \) defining the PMNS establishes a one-to-one correspondence between the polynomial \( A(x) \) from the range \( P(x) \) and the polynomial modular code

\[(a_1(x), a_2(x), \ldots, a_n(x))\]

where the \( l \)-th component is the residual \( a_l(x) = \langle A(x) \rangle_{p_l(x)} \) of division of \( A(x) \) by a module \( p_l(x) \) \( (l = 1, 2, \ldots, n) \). The ring operations in the PMNS
over any two polynomials

\[ A(x) = (a_1(x), a_2(x), \ldots, a_n(x)) \quad \text{and} \quad B(x) = (b_1(x), b_2(x), \ldots, b_n(x)) \]

are naturally executed independently on each of residues, i.e. according to the rule

\[ \langle A(x) \circ B(x) \rangle = \left( \langle a_1(x) \circ b_1(x) \rangle_{p_1(x)}, \langle a_2(x) \circ b_2(x) \rangle_{p_2(x)}, \ldots, \langle a_n(x) \circ b_n(x) \rangle_{p_n(x)} \right), \quad (3) \]

where \( \circ \in \{+, \cdot\} \).

As long as in the PMNS all operations (both modular and non-modular) are performed in the ring \( \mathbb{Z}_m \), and this ring is included in the conditional notation (2). This ring is called the scalar range or the numeric range of the PMNS. It follows from formula (3) that the efficiency level of the PMNS arithmetic depends significantly on the degrees \( \deg p_l(x) \) of modules \( p_l(x) \), and its analytical form, on the one hand, and on the number system in which calculation over polynomial residuals in the ring \( \mathbb{Z}_m \) are performed, on the other hand. Due to the modular structure of these calculations it is quite natural to use the real MNS with the modules \( m_1, m_2, \ldots, m_k \) for encoding and processing of elements from the scalar range \( \mathbb{Z}_m \) [2, 3]. With such an approach the parameter \( m \) is equal to \( M_k = \prod_{i=1}^{k} m_i \), i.e. the ring

\[ \mathbb{Z}_{M_k} = \{0, 1, \ldots, M_k - 1\} \]

is used as a numeric range of the PMNS.

**Definition 4.** The PMNS with modular coding of elements of scalar range is called the polynomial-numerical or polynomial-scalar MNS with the modules \( (p_1(x), p_2(x), \ldots, p_n(x)), m_1, m_2, \ldots, m_k \) and is denoted by the symbolic notation \( \langle \langle \cdot \rangle_{P(x)}; | \cdot |_{M_k} \rangle \).

As for the problem of a choice of polynomial modules

\[ (p_1(x), p_2(x), \ldots, p_n(x)) \]

it is clear that the normalized polynomials of the first degree \( p_l(x) = x - r_l \), \( (r_l \in \mathbb{Z}_{M_k} \quad l = 1, 2, \ldots, n) \) are the most appropriate. At the same time for
practical applications, for example, in the digital signal processing, digital communications or coding theory the polynomial modules for which

\[ P(x) = \prod_{l=1}^{n} p_l(x) \]

is of the form \( x^n \pm 1 \) are of particular interest. It was shown in several studies that the indicated restrictions to the choice of modules PMNS are compatible.

In the rings \( \mathbb{Z}_{Mk} \) the polynomials \( x^n \pm 1 \) admit a factorization of the form

\[ x^n \pm 1 = \prod_{l=1}^{n} (x - r_l). \]

4. MINIMAL REDUNDANT POLYNOMIAL MODULAR NUMBER SYSTEM

The principle of minimal redundant modular coding assumes that the set

\[ \mathbb{Z}_{2M} = \{ \lfloor -M \rfloor, \lfloor -M + 1 \rfloor, \ldots, \lfloor M - 1 \rfloor \} \]

\((M = \prod_{i=0}^{k-1} m_i, \ m_0 \) is the auxiliary natural module\) is used as a scalar range of the PMNS \([2, 3]\).

Definition 5. The PMNS with minimal redundant modular coding of the elements of a scalar range is called the minimal redundant polynomial-numerical or polynomial-scalar MNS with the modules

\[(p_1(x), p_2(x), \ldots, p_n(x)), m_1, m_2, \ldots, m_k\]

and is denoted by the symbolic notation \( \langle \cdot \rangle_{P(x)}; \lfloor \cdot \rfloor_{2M} \).

Let us consider the minimal redundant polynomial-scalar MNS (PSMNS) with modules \( m_1, m_2, \ldots, m_k \) and \( p_l(x) = x - r_l \) \((r_l \in \mathbb{Z}_{2M})\) such that \( P(x) = x^n - 1 \) or \( P(x) = x^n + 1 \). In accordance with the stated above, an arbitrary polynomial \( A(x) \in \langle \cdot \rangle_{P(x)} \) in minimal redundant PSMNS is encoded by a set of residues

\[(A_{1,1}, A_{1,2}, \ldots, A_{1,k}; A_{2,1}, A_{2,2}, \ldots, A_{2,k}; \ldots; A_{n,1}, A_{n,2}, \ldots, A_{n,k}) \quad (4)\]

where \( A_{l,i} = |A_l|_{m_i}; \ A_l = \langle A(x) \rangle_{p_l(x)} \) is a residue of division \( A(x) \) by a module \( p_l(x) = x - r_l \) which taking into account the Bezout theorem \([8]\), is calculated by the formula \( A_l = |A(r_l)|_{M_k}; \ l = 1, 2, \ldots, n; \ i = 1, 2, \ldots, k. \)
Decoding mapping for minimal redundant PSMNS \(\langle \cdot \rangle_{P(x)}; |\cdot|_{2M}\) assigning to each code of the form (4) a polynomial \(A(x) = \sum_{\nu=0}^{n-1} a_\nu x^\nu\) from the range \(\langle \cdot \rangle_{P(x)}\) is implemented by means of the following theorem.

**Theorem 1.** If \(p_l(x) = x - r_l\) \((r_l \in \mathbb{Z}_{M_k})\), \(P(x) = \prod_{l=1}^{n} p_l(x) = x^n \pm 1\) and \((n, M_k) = 1\), then the coefficients of the polynomial \(A(x) = \sum_{\nu=0}^{n-1} a_\nu x^\nu\) in \(\langle \cdot \rangle_{P(x)}\) corresponding to a minimal redundant position-scalar modular code (4) are defined by the following relations

\[
a_\nu = \sum_{i=1}^{k-1} M_{i,k-1}^{-1} M_{i,k-1}^{-1} a_\nu |a_\nu|_{m_i} + I(a_\nu) M_{k-1},
\]

where \(I(a_\nu)\) is an interval index of the number \(a_\nu\) calculated in accordance with the relations given in the article [2].

The validity of the use of interval-modular form (5) to restore the values of the polynomial coefficients \(a_\nu\) on the basis of their modular code

\(|a_\nu|_{m_1}, |a_\nu|_{m_2}, \ldots, |a_\nu|_{m_k}\)

is guaranteed by the minimum redundancy of the encoding elements of a scalar range \(\mathbb{Z}_{2M}\). As for the formula (6), it follows from the Chinese remainder theorem [1, 9] which for the PMNS with modules \(p_1(x), p_2(x), \ldots, p_n(x)\) gives

\[
A(x) = \sum_{l=1}^{n} P_l(x) \langle P_l(x)^{-1} A(x) \rangle_{p_l(x)} = \sum_{l=1}^{n} P_l(x) \langle P_l(x)^{-1} A_l \rangle_{p_l(x)},
\]

where \(P_l(x) = P(x)/p_l(x) = (x^n \pm 1)/(x - r_l)\).

The ring operations in the PSMNS are performed component-wise. In accordance with (3), the operation \(\circ \in \{+, \cdot\}\) over any two polynomials \(A(x)\) and \(B(x)\) is executed by the rule
\[(A_{1,1}, A_{1,2}, \ldots, A_{1,k}; A_{2,1}, A_{2,2}, \ldots, A_{2,k}; \ldots; A_{n,1}, A_{n,2}, \ldots, A_{n,k}) \circ \circ (B_{1,1}, B_{1,2}, \ldots, B_{1,k}; B_{2,1}, B_{2,2}, \ldots, B_{2,k}; \ldots; B_{n,1}, B_{n,2}, \ldots, B_{n,k}) = \\
= (|A_{1,1} \circ B_{1,1}|_{m_1}, |A_{1,2} \circ B_{1,2}|_{m_2}, \ldots, |A_{1,k} \circ B_{1,k}|_{m_k}; \\
|A_{2,1} \circ B_{2,1}|_{m_1}, |A_{2,2} \circ B_{2,2}|_{m_2}, \ldots, |A_{2,k} \circ B_{2,k}|_{m_k}; \ldots \\
|A_{n,1} \circ B_{n,1}|_{m_1}, |A_{n,2} \circ B_{n,2}|_{m_2}, \ldots, |A_{n,k} \circ B_{n,k}|_{m_k}),
\]

where \(A_{l,i} = |A(r_l)|_{m_i}\) and \(B_{l,i} = |B(r_l)|_{m_i}\) are the digits of polynomial-scalar modular codes of the operands \(A(x)\) and \(B(x)\), respectively (see (4)).

The unique possibility to calculate the sum, difference and especially the product of two polynomials in accordance with (8) in one clock tick is one of the main advantages of the PSMNS. Thus, in this system both the addition and the multiplication of any two polynomials modulo \(P(x) = x^n \pm 1\) for their implementation require \(n\) real additions and multiplications, respectively, which can also execute in parallel. In contrast, in the case of traditional arithmetic the computational complexity of the two polynomials multiplication in the ring \(\langle \cdot \rangle_{P(x)}\) amounts to \(n(n-1)\) real additions and \(n^2\) real multiplications.

The minimal redundant PMNS have all advantages of the classical PMNS. In addition, these algebraic systems are characterized by more simple computer arithmetic. The use of minimal redundant coding on the lower level allows us to increase the efficiency of computer arithmetic due to optimization of the non-modular procedures [1]. Moreover, it should be noted that on the basis of technique for constructing the MNS [2, 3] the minimal redundant PSMNS with a range of complex scalars can be defined. In this case the efficiency gain is even more impressive than in the case of real PSMNS.

References


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CONTINUITY OF SUPERQUADRATIC SET-VALUED FUNCTIONS

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Abstract

Let \( X = (X,+ ) \) be an arbitrary topological group. The aim of the paper is to prove a regularity theorem for set-valued superquadratic functions, that is solutions of the inclusion

\[
2F(s) + 2F(t) \subset F(s + t) + F(s - t), \quad s, t \in X,
\]

with values in a topological vector space.

1. Introduction

In the present paper superquadratic set-valued functions, defined on a topological group \( X \), that is solutions of the inclusion

(1) \[
2F(s) + 2F(t) \subset F(s + t) + F(s - t), \quad s, t \in X,
\]

with non-empty, compact and convex values in a topological vector space are studied. If the sign of the inclusion in (1) is replaced by “\( \supset \)” then \( F \) is called subquadratic set-valued function and if we have “\( = \)” instead of “\( \subset \)” in (1) then we say that \( F \) is quadratic set-valued function. A regularity theorem for a subquadratic set-valued function \( F \), which was considered in [7], stating that upper semi-continuity at a point zero with condition \( F(0) = \{0\} \) implies the continuity of a subquadratic set-valued function \( F \) everywhere in \( X \). Now, we investigate a regularity theorem for superquadratic set-valued functions.

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It is proved here that lower semi-continuity at a point zero implies the continuity of a superquadratic set-valued function, which has the property \((O)\), everywhere in \(X\). In the case of single real valued subquadratic functions the continuity problem is considered in [2]. For functions of this kind properties of subquadratic and superquadratic functions are quite analogous, in view of the fact, if a function \(f\) is subquadratic then the function \(-f\) is superquadratic and conversely. It is not necessary to investigate functions of these two kinds individually. In the case of set-valued functions the situation is different. Namely, some properties of subquadratic set-valued functions do not have their analogs for superquadratic set-valued functions and conversely. Moreover, even if properties of subquadratic and superquadratic set-valued functions are similar we have to prove them separately. This is the reason why both functions of this kind are considered. Like in the case of subquadratic set-valued functions, the regularity theorem for superquadratic set-valued functions generalizes some earlier results of this type obtained by D. Henney [1], K. Nikodem [3] and W. Smajdor [6] for quadratic set-valued functions. We start our consideration from basic properties for superquadratic set-valued functions, which play an important role in the proof of the main theorem, which is presented in the third part.

Let us start with the notation used in this paper. Throughout this paper \(\mathbb{R}\) stands for the set of reals. Let \(X\) be a topological group and \(Y\) be a topological vector space. Let \(n(Y)\) denotes the family of all non-empty subsets of \(Y\), \(c(Y)\) — the family of all compact members of \(n(Y)\), \(cc(Y)\) — the family of all convex members of \(c(Y)\) and \(B(Y)\) — the family of all subsets of \(n(Y)\). The term set-valued function will be abbreviated in the form s.v.f.

Now we present here some definitions for the sake of completeness.

**Definition 1.1.** (cf. [6]) A s.v.f. \(F : X \to n(Y)\) is said to be upper semi-continuous (abbreviated u.s.c.) at \(x \in X\) iff for every neighbourhood \(V\) of zero in \(Y\) there exists a neighbourhood \(U\) of zero in \(X\) such that

\[ F(x + t) \subset F(x) + V \]

for every \(t \in U\).

**Definition 1.2.** (cf. [6]) A s.v.f. \(F : X \to n(Y)\) is said to be lower semi-continuous (abbreviated l.s.c.) at \(x \in X\) iff for every neighbourhood \(V\) of zero
in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that
\[ F(x) \subset F(x + t) + V \]
for every $t \in U$.

**Definition 1.3.** (cf. [6]) A s.v.f. $F: X \to n(Y)$ is said to be continuous at $x \in X$ iff it is both u.s.c. and l.s.c. at $x$. It is said to be continuous iff it is continuous at every point $x \in X$.

**Definition 1.4.** (cf. [6]) Let $X$ be a set and $Y$ be a real topological vector space. A s.v.f. $F: X \to n(Y)$ is said to be bounded on a set $B \subset X$ iff the set
\[ F(B) := \bigcup \{ F(t) | t \in B \} \]
is bounded in $Y$.

**Definition 1.5.** (cf. [6]) Let $X$ be a set and $Y$ be a real topological vector space. A s.v.f. $F: X \to n(Y)$ is said to be bounded at a point $x_0 \in X$ iff there exists a neighbourhood $U$ of zero in $X$ such that $F$ is bounded on $x_0 + U$.

We adopt the following two definitions.

**Definition 1.6.** Let $X$ be a topological group. A set $A \subset X$ is bounded in $X$ iff for every neighbourhood $U$ of zero in $X$ there exists an $n \in \mathbb{N} \cup \{0\}$ such that
\[ A \subset 2^n U. \]

**Definition 1.7.** Let $X$ be a topological group and $Y$ be a vector space. A s.v.f. $F: X \to n(Y)$ has the property $(O)$ iff for every bounded set $A$ in $X$ the set $F(A)$ is bounded in $Y$.

We will use frequently the following well known lemma.

**Lemma 1.8.** (see [5]) Let $Y$ be a topological vector space. Let $A, B, C$ be subsets of $Y$ such that $A + C \subset B + C$. If $B$ is closed and convex and $C$ is bounded then $A \subset B$.

In our proofs we will often use three known lemmas (see Lemma 1.1, Lemma 1.3 and Lemma 1.6 in [3]). The first lemma says that for a convex subset $A$ of an arbitrary real vector space $Y$ the equality $(s + t)A = sA + tA$ holds for every $s, t \geq 0$ (or $s, t \leq 0$). The second lemma says that for two convex
subsets $A, B \subseteq Y$ the set $A + B$ is also convex and the last lemma says that if $A \subseteq Y$ is a closed set and $B \subseteq Y$ is a compact set then the set $A + B$ is closed.

2. Basic properties

Let us start with definition.

**Definition 2.1.** A topological group $X$ is said to be locally bounded group iff there exists in it a bounded neighbourhood of zero.

**Lemma 2.2.** Let $X$ be a group and $Y$ be a real topological vector space. If for a superquadratic s.v.f. $F : X \to n(Y)$ the set $F(0) \in B(Y)$, then $F(0) = \{0\}$.

*Proof.* Putting $t = s = 0$ in (1) and using Lemma 1.1 in [3], we get

$$F(0) + F(0) \supseteq 2F(0) + 2F(0) = 2(F(0) + F(0)).$$

Repeating it $n$-times, we obtain

$$(2) \quad F(0) + F(0) \supseteq 2^n (F(0) + F(0)).$$

In spite of the fact, that the sum of two bounded sets is also bounded, the set $F(0) + F(0)$ is bounded. By boundedness of the set $F(0) + F(0)$ and by (2), we get

$$F(0) + F(0) = \{0\}.$$

Hence, $F(0) = \{0\}$. $\square$

**Lemma 2.3.** Let $X$ be a group and $Y$ be a real topological vector space. If an s.v.f. $F : X \to n(Y)$ with bounded, closed and convex values is superquadratic, then

$$(3) \quad n^2 F(x) \subset F(nx),$$

for every $x \in X$ and $n \in \mathbb{N}$.

*Proof.* According to Lemma 2.2 $F(0) = \{0\}$. The proof of the inclusion (3) we can obtain in the same way as the proof of Lemma 2.2 in [7] for subquadratic s.v.f. It is sufficient to replace the sign “$\subset$” in the proof of Lemma 2.2 in [7] by the sign “$\supset$”. $\square$
Lemma 2.4. Let \( X \) be a topological group and \( Y \) be a real topological vector space. If a s.v.f. \( F : X \to n(Y) \) is superquadratic and bounded on a neighbourhood of a point \( x_0 \in X \), then \( F \) is bounded on a neighbourhood of zero in \( X \).

Proof. Let \( V_0 \) be a neighbourhood of zero in \( Y \) and let \( U_0 \) be a symmetric neighbourhood of zero in \( X \) such that the set \( F(x_0 + U_0) \) is bounded. Let \( V \) be a symmetric neighbourhood of zero in \( Y \) such that

\[
V + V + V + V \subset V_0.
\]

There exists a \( c > 0 \) such that

\[
cF(x_0 + U_0) \subset V.
\]

Setting \( s = x_0 \) in (1) and taking arbitrary \( t \in U_0 \), we get

\[
2F(t) + 2F(x_0) \subset F(x_0 + t) + F(x_0 - t).
\]

Fix an \( a \in 2F(x_0) \). Then

\[
2F(t) \subset F(x_0 + t) + F(x_0 - t) - a \subset F(x_0 + t) + F(x_0 - t) - 2F(x_0),
\]

for every \( t \in U_0 \). By inclusions (4)-(6) and Lemma 2.3, we get

\[
2cF(t) \subset cF(x_0 + t) + cF(x_0 - t) - cF(x_0) - cF(x_0) \subset V + V + V + V \subset V_0
\]

for every \( t \in U_0 \). Thus \( F \) is bounded on \( U_0 \) and the proof is ended.

Lemma 2.5. Let \( X \) be a 2-divisible topological group and \( Y \) be a real topological vector space. If a s.v.f. \( F : X \to n(Y) \) is superquadratic with closed, bounded, convex values and bounded on a neighbourhood of a point \( x \in X \), then it is u.s.c. at zero in \( X \).

Proof. The proof of this lemma we can obtain likewise as the proof of Lemma 4.7 in [6] for quadratic set-valued functions. Let \( V \) be an arbitrary neighbourhood of zero in \( Y \). By Lemma 2.4 there exists a neighbourhood \( U \) of zero in \( X \) such that \( F \) is bounded on \( U \). Hence, there exists an \( n \in \mathbb{N} \) such that

\[
\frac{1}{4^n}F(U) \subset V.
\]

According to Lemma 2.3, we get

\[
F\left(\frac{1}{2^n}U\right) \subset \frac{1}{4^n}F(U).
\]
Consequently

(9) \[ F\left(\frac{1}{2^n}U\right) \subset V. \]

By Lemma 2.2 \( F(0) = \{0\} \). Since

\[ F\left(\frac{1}{2^n}U\right) = \bigcup \left\{ F(t) \mid t \in \frac{1}{2^n}U \right\}, \]

\( F(0) = \{0\} \) and the inclusion (9) holds, we get

\[ F(t) \subset V + F(0), \]

for every \( t \in \frac{1}{2^n}U \), which means that the s.v.f. \( F \) is upper semicontinuous at zero. \( \square \)

The following corollary follows by Lemma 2.5.

**Corollary 2.6.** Let \( X \) be a 2-divisible locally bounded topological group and \( Y \) be a real topological vector space. If a s.v.f. \( F: X \to n(Y) \) is superquadratic with closed, bounded, convex values and has the property \((O)\), then it is u.s.c. at zero.

**Proof.** Let \( x_0 \in X \) and let \( U \) be a bounded neighbourhood of zero in \( X \). The set \( F(x_0 + U) \) is bounded since \( F \) has property \((O)\).

Then s.v.f. \( F \) is u.s.c. at zero according to Lemma 2.5. \( \square \)

3. The main result

The proof of the next lemma is similar to the proof of Theorem 3.2 in [7], which is a regularity theorem for subquadratic set-valued functions.

**Lemma 3.1.** Let \( X \) be a 2-divisible locally bounded topological group and \( Y \) be a locally convex real topological space. If a superquadratic s.v.f. \( F: X \to cc(Y) \) has property \((O)\), then it is upper semicontinuous everywhere in \( X \).

**Proof.** Similar as in the proof of Theorem 3.2 in [7], the idea of the proof of this lemma is due to W. Smajdor (Lemma 4.8 in [6]). Suppose that s.v.f. is not u.s.c. at \( z \in X \). Then there exists a neighbourhood \( V \) of zero in \( Y \) such that for every neighbourhood \( U \) of zero in \( X \) there exists \( x_u \in U \) such that

\[ F(z + x_u) \not\subset F(z) + V. \]
Take a convex balanced neighbourhood $W$ of zero in $Y$ such that $W \subset V$. Then
\[(10) \quad F(z + x_u) \not\subset F(z) + W.\]

By induction we shall show that
\[(11) \quad F(z + 2^k x_u) \not\subset F(z) + 2^k \left(2^k - 1\right) F(x_u) + 2^k W\]
for $k = 1, 2, \ldots$. For $k = 0$ (11) holds by (10). Now, assume that (11) holds for some positive integer $k \geq 0$. By (1), we have
\[
2F(z + 2^k x_u) + 2F(2^k x_u) \subset F(z + 2^k x_u + 2^k x_u) + F(z + 2^k x_u - 2^k x_u) = F(z + 2^{k+1} x_u) + F(z).
\]

By Lemma 2.3
\[
2F(z + 2^k x_u) + 2^{2k+1} F(x_u) \subset 2F(z + 2^k x_u) + 2F(2^k x_u).
\]

Consequently
\[(12) \quad 2F(z + 2^k x_u) + 2^{2k+1} F(x_u) \subset F(z + 2^{k+1} x_u) + F(z)\]

According to (11) and Lemma 1.8, we obtain
\[(13) \quad 2F(z + 2^k x_u) + 2^{2k+1} F(x_u) \not\subset 2F(z) + 2^{k+1} \left(2^k - 1\right) F(x_u) + 2^{k+1} W + 2^{2k+1} F(x_u).\]

By (12) and (13), we have
\[
F(z + 2^{k+1} x_u) + F(z) \not\subset 2F(z) + 2^{k+1} \left(2^k - 1\right) F(x_u) + 2^{k+1} W + 2^{2k+1} F(x_u).
\]

By convexity of the set $F(x_u)$ and by Lemma 1.1 in [6], we get
\[
2F(z) + 2^{k+1} \left(2^k - 1\right) F(x_u) + 2^{k+1} W + 2^{2k+1} F(x_u) =
\]
\[
= 2F(z) + [2^k \left(2^k - 1\right) + 2^{2k+1}] F(x_u) + 2^{k+1} W.
\]

Applying the equality
\[
2^{k+1} \left(2^k - 1\right) + 2^{2k+1} = 2^{k+1} \left(2^{k+1} - 1\right),
\]
we obtain
\[
F(z + 2^{k+1} x_u) + F(z) \not\subset 2F(z) + 2^{k+1} \left(2^{k+1} - 1\right) F(x_u) + 2^{k+1} W.
\]
By convexity of the set $F(z)$ and by Lemma 1.1 in [6], we get finally

$$F(z + 2^{k+1}x_u) \notin F(z) + 2^{k+1} \left(2^{k+1} - 1\right) F(x_u) + 2^{k+1}W.$$  

Thus (11) is generally valid for all integer $k \geq 0$.

Take a bounded set $U_0$ of zero in $X$. Since $F$ has property $(O)$, there exists $\lambda > 0$ such that

$$\lambda F(z + x) \subset W, \quad x \in U_0. \quad (14)$$

Now we choose an $k \in \mathbb{N}$ so large that the inequality

$$2^k > \frac{3}{\lambda} \quad (15)$$

holds. By Corollary 2.6 $F$ is u.s.c at zero and $F(0) = \{0\}$ according to Lemma 2.2. Since $F$ is u.s.c. at zero and $F(0) = \{0\}$ there exists a neighbourhood $U$ of zero in $X$ such that

$$F(t) \subset \frac{1}{\lambda 2^k(2^k - 1)} W, \quad t \in U$$

and

$$U \subset \frac{1}{2^k} U_0.$$

Let $x_u \in U$ satisfies condition (11). Moreover

$$2^k x_u \in U_0 \quad (16)$$

and

$$2^k(2^k - 1)F(x_u) \subset \frac{1}{\lambda} W. \quad (17)$$

Let $a \in F(z + 2^k x_u)$, $b \in F(z)$ and $c \in F(x_u)$. Then

$$a = b + \left(a - b - 2^k(2^k - 1)c\right) + 2^k(2^k - 1)c.$$

By inclusions (14), (16), (17) and (15), we obtain

$$a - b - 2^k(2^k - 1)c \in \frac{1}{\lambda} W + \frac{1}{\lambda} W + \frac{1}{\lambda} W = \frac{3}{\lambda} W \subset 2^k W.$$

Therefore

$$a \in F(z) + 2^k W + 2^k(2^k - 1)F(x_u)$$

and

$$F\left(z + 2^k x_u\right) \subset F(z) + 2^k W + 2^k(2^k - 1)F(x_u),$$

in spite of (11), which ends the proof. \qed
The following theorem is the main result of this paper.

**Theorem 3.2.** Let $X$ be a 2-divisible locally bounded topological group and let $Y$ be a locally convex real topological space. If a superquadratic s.v.f. $F: X \to \text{cc}(Y)$ is l.s.c. at zero and has property (O), then it is continuous everywhere in $X$.

*Proof.* By Lemma 3.1 $F$ is u.s.c. everywhere in $X$. Now, we show that $F$ is l.s.c. in $X$. Let $x_0 \in X$ and let $V$ be a neighbourhood of zero in $Y$. We choose convex neighbourhood $V_0$ of zero in $Y$ such that $3V_0 \subset V$. Since $F$ is u.s.c. at $x_0$ there exists a symmetric neighbourhood $U$ of zero in $X$ such that

\[(18) \quad F(x_0 + t) \subset F(x_0) + V_0,\]
\[(19) \quad F(x_0 - t) \subset F(x_0) + V_0\]

if $t \in U$.

According to lemma 2.2 $F(0) = \{0\}$. Since $F$ is l.s.c. at zero and $F(0) = \{0\}$, there exists a neighbourhood $U_0$ of zero in $X$ such that

\[(20) \quad 0 \in F(t) + V_0 \quad t \in U_0.\]

Let $\tilde{U}$ be a symmetric neighbourhood of zero in $X$, such that $\tilde{U} \subset U \cap U_0$.

Now, let $t \in \tilde{U}$. By (1), (19), (20) and by the fact $F(0) = \{0\}$, we obtain

\[F(x_0) + \{0\} \subset F(x_0) + F(t) + V_0 \subset \frac{1}{2}F(x_0 + t) + \frac{1}{2}F(x_0 - t) + V_0 \subset \]
\[\subset \frac{1}{2}F(x_0 + t) + \frac{1}{2}F(x_0) + \frac{3}{2}V_0.\]

By convexity of the set $F(x_0)$ and by Lemma 1.1 in [3], we get

\[\frac{1}{2}F(x_0) + \frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0) + \frac{1}{2}F(x_0 + t) + \frac{3}{2}V_0.\]

Since the set $\frac{1}{2}F(x_0)$ is bounded and the set $\frac{1}{2}F(x_0 + t) + \frac{3}{2}V_0$ is convex (see Lemma 1.3 in [3]) and closed (Lemma 1.6 in [3]), then according to Lemma 1.8 we have proved that

\[\frac{1}{2}F(x_0) \subset \frac{1}{2}F(x_0 + t) + \frac{3}{2}V_0.\]

Therefore,

\[F(x_0) \subset F(x_0 + t) + 3V_0 \subset F(x_0 + t) + V.\]

Thus $F$ is l.s.c in $X$. The proof is completed. \(\square\)
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THE STABILITY OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

IWONA TYRALA

ABSTRACT

In the present paper we deal with the Dhombres-type trigonometric difference

\[ f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)f(y) \]

assuming that its absolute value is majorized by some constant. Our aim is to find functions \( \tilde{f} \) and \( \tilde{g} \) which satisfy the Dhombres-type trigonometric functional equation and for which the differences \( \tilde{f} - f \) and \( \tilde{g} - g \) are uniformly bounded.

1. Introduction

Stability problems concerning classical functional equations have been treated by several authors (see, e.g., [5]–[7]). The cosine functional equation

(1) \[ f(x+y) + f(x-y) = 2f(x)f(y) \]

and the sine functional equation

(2) \[ f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \]

are both stable (in fact, they are even superstable) in the Hyers-Ulam sense. In [3], Baker studied the stability of the cosine functional equation (1), while Cholewa established the stability of the sine functional equation (2) in [4].
results about the superstability can be obtained as corollaries from theorems by Badora and Ger. Namely, the following theorems hold.

**Theorem 1** (Badora and Ger, see [2]). Let \((G, +)\) be an Abelian group and let \(f: G \to \mathbb{C}\) and \(\varphi: G \to \mathbb{R}\) satisfy the inequality
\[
|f(x + y) + f(x - y) - 2f(x)f(y)| \leq \varphi(x) \quad \text{for all } x, y \in G.
\]
Then either \(f\) is bounded or
\[
f(x + y) + f(x - y) = 2f(x)f(y) \quad \text{for all } x, y \in G.
\]

**Theorem 2** (Badora and Ger, see [2]). Let \((G, +)\) be a uniquely 2-divisible Abelian group and let \(f: G \to \mathbb{C}\) and \(\varphi: G \to \mathbb{R}\) satisfy the inequality
\[
|f(x)f(y) - f\left(\frac{x + y}{2}\right)^2 + f\left(\frac{x - y}{2}\right)^2| \leq \varphi(x) \quad \text{for all } x, y \in G.
\]
Then either \(f\) is bounded or
\[
f(x)f(y) = f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2 \quad \text{for all } x, y \in G.
\]

From now on, we denote the odd and the even parts of a function \(f\) by \(f_o\) and \(f_e\), respectively. The next lemma (due to Wilson, see [11]) provides general solutions of an equation that generalizes the equation (1).

**Lemma 1** (see also [1] and [8]). Let \((G, +)\) be an Abelian group. Then functions \(f, g: G \to \mathbb{C}\) satisfy the functional equation
\[
(3) \quad f(x + y) + f(x - y) = 2f(x)g(y)
\]
if and only if one of the following conditions holds:

(i) the function \(g\) is arbitrary and \(f = 0\);

(ii) there exist an additive function \(a: G \to \mathbb{C}\) and a constant \(\alpha \in \mathbb{C}\) such that
\[
f(x) = a(x) + \alpha \quad \text{and} \quad g(x) = 1 \quad \text{for all } x \in G;
\]

(iii) there exist an exponential function \(m: G \to \mathbb{C}\) and constants \(\beta, \gamma \in \mathbb{C}\) such that
\[
f(x) = \beta m_o(x) + \gamma m_e(x) \quad \text{and} \quad g(x) = m_e(x) \quad \text{for all } x \in G.
\]
THE STABILITY OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

In [8], Székelyhidi studied the Hyers-Ulam stability of the equation (3), obtaining the following result.

**Theorem 3.** Let \((G, +)\) be an Abelian group and let \(\varepsilon \geq 0\). If functions \(f, g: G \to \mathbb{C}\) satisfy the inequality
\[
|f(x + y) + f(x - y) - 2f(x)g(y)| \leq \varepsilon \quad \text{for all } x, y \in G,
\]
then one of the following conditions holds:

(i) if \(f = 0\), then \(g\) is arbitrary;
(ii) if \(f \neq 0\) is bounded, then \(g\) is bounded, as well;
(iii) if \(g\) is bounded and \(f\) is unbounded, then \(g = 1\) and there exist an additive function \(A: G \to \mathbb{C}\) and a constant \(\delta \in \mathbb{C}\) such that
\[
|f(x) - A(x)| \leq \delta \quad \text{for all } x \in G;
\]
(iv) if \(f \neq 0\) and \(g\) is unbounded, then \(f\) is unbounded, as well. Moreover, functions \(f\) and \(g\) satisfy the equation (3).

The above theorems allow us to formulate the following result concerning superstability.

**Corollary 1.** Let unbounded functions \(f, g: G \to \mathbb{C}\) satisfy the inequality
\[
|f(x + y) + f(x - y) - 2f(x)g(y)| \leq \varepsilon
\]
for all \(x, y \in G\) and for some \(\varepsilon \geq 0\). Then \(f\) and \(g\) satisfy the equation (3).

The aim of this paper is to study stability properties of the Dhombres-type trigonometric functional equation, i.e.,
\[
f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2 + f(x + y) + f(x - y) = f(x)[f(y) + g(y)].
\]
In the case when \(g = 2h\), solutions of the above equation can be found in [10].

We shall use the following lemma.

**Lemma 2** (see [9, Corollary 3]). Let \((G, +)\) be a uniquely 2-divisible Abelian group and let \(\varepsilon \geq 0\). Let an unbounded function \(f: G \to \mathbb{C}\) and a function \(g: G \to \mathbb{C}\) satisfy the inequality
\[
|f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2 - f(x)g(y)| \leq \varepsilon \quad \text{for all } x, y \in G.
\]
Then one of the following conditions holds:
(i) if \( g \neq 0 \) is bounded, then \( g \) satisfies the sine equation (2);
(ii) if \( g \) is unbounded, then there exists a function \( h: G \to \mathbb{C} \) such that
\[
f(x + y) + f(x - y) = 2f(x)h(y) \quad \text{for all } x, y \in G.
\]
For \( f = f_\varepsilon + f_o \), we have \( f_\varepsilon(x) = f(0)h(x) \) for all \( x \in G \) and \( f_o \) satisfies the sine equation. Moreover, if \( f(0) = 0 \), then \( g = f = f_o \).

2. Main results

Our main result reads as follows.

**Theorem 4.** Let \((G, +)\) be a uniquely 2-divisible Abelian group and let \( \varepsilon \geq 0 \). If functions \( f, g: G \to \mathbb{C} \) satisfy the inequality
\[
\left| f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 + f(x+y) + f(x-y) - f(x)[f(y) + g(y)] \right| \leq \varepsilon
\]
for all \( x, y \in G \), then there exist an exponential function \( m: G \to \mathbb{C} \), additive functions \( a, A: G \to \mathbb{C} \), a bounded function \( B: G \to \mathbb{C} \) and a constant \( \beta \) such that one of the following conditions holds:

(i) if \( f = 0 \), then \( g \) is arbitrary;
(ii) if \( f \neq 0 \) is bounded, then \( g \) is bounded, as well;
(iii) if the function \( f \) is unbounded, then
\[
\begin{cases}
  f(x) = A(x) + B(x) \\
  g(x) = a(x) - A(x) - B(x) + 2
\end{cases}
\quad \text{for all } x \in G,
\]
or
\[
\begin{cases}
  f(x) = f_o(x) + f(0)m_e(x) \\
  g(x) = (\beta m_o(x) - f_o(x)) + (2 - f(0))m_e(x)
\end{cases}
\quad \text{for all } x \in G.
\]

Moreover, suppose that \( f(0) = 0 \). Then
\[
\begin{cases}
  f(x) = \beta m_o(x) \\
  g(x) = 2m_e(x)
\end{cases}
\quad \text{for all } x \in G.
\]

**Proof.** Assume that the function \( f \) is an unbounded solution of inequality (5). Then there exists a sequence \((z_n)_{n \in \mathbb{N}}\) of elements of \( G \) such that
\[
0 \neq |f(z_n)| \to \infty \quad \text{as } n \to \infty.
\]
Let us take \( x = z_n \) in (5). Then we obtain
\[
\left| f\left( \frac{z_n + y}{2} \right)^2 - f\left( \frac{z_n - y}{2} \right)^2 + f(z_n + y) + f(z_n - y) - f(z_n)\right| \leq \varepsilon
\]
for all \( y \in G \) and \( n \in \mathbb{N} \), whence
\[
\left| f\left( \frac{z_n + y}{2} \right)^2 - f\left( \frac{z_n - y}{2} \right)^2 + f(z_n + y) + f(z_n - y) - \frac{f(y) + g(y)}{f(z_n)} \right| \leq \frac{\varepsilon}{|f(z_n)|}
\]
for all \( y \in G \) and \( n \in \mathbb{N} \). Now, taking the limit as \( n \to \infty \) and applying (9), we obtain
\[
\lim_{n \to \infty} \frac{f\left( \frac{z_n + y}{2} \right)^2 - f\left( \frac{z_n - y}{2} \right)^2 + f(z_n + y) + f(z_n - y)}{f(z_n)} = f(y) + g(y)
\]
for all \( y \in G \). Hence,
\[
f(0) + g(0) = 2.
\]

Let us replace \( x \) by \( z_n + x \) in (5). Then we get
\[
\left| f\left( \frac{z_n + x + y}{2} \right)^2 - f\left( \frac{z_n + x - y}{2} \right)^2 + f(z_n + x + y) + f(z_n + x - y) - f(z_n + x)\right| \leq \varepsilon.
\]
Similarly, let us replace \( x \) by \( z_n - x \) in (5). Then
\[
\left| f\left( \frac{z_n - x + y}{2} \right)^2 - f\left( \frac{z_n - x - y}{2} \right)^2 + f(z_n - x + y) + f(z_n - x - y) - f(z_n - x)\right| \leq \varepsilon.
\]

From the above inequalities we compute
\[
\left| f\left( \frac{z_n + (x + y)}{2} \right)^2 - f\left( \frac{z_n - (x + y)}{2} \right)^2 + f(z_n + (x + y)) + f(z_n - (x + y))\right| \leq \varepsilon
\]
\[
+ f(z_n - (x + y)) + f\left( \frac{z_n + (-x + y)}{2} \right)^2 - f\left( \frac{z_n - (-x + y)}{2} \right)^2
\]
\[
+ f(z_n + (-x + y)) + f(z_n - (-x + y))
\]
\[
- \left[ f(z_n + x) + f(z_n - x) \right] \cdot [f(y) + g(y)] \leq \varepsilon
\]
for all \( x, y \in G \) and \( n \in \mathbb{N} \). Therefore,
\[
\left| f \left( \frac{z_n + (x+y)}{2} \right)^2 - f \left( \frac{z_n - (x+y)}{2} \right)^2 + f(z_n + (x+y)) + f(z_n - (x+y)) \right|
\]
\[
\left| \frac{f \left( \frac{z_n + (-x+y)}{2} \right)^2 - f \left( \frac{z_n - (-x+y)}{2} \right)^2 + f(z_n + (-x+y)) + f(z_n - (-x+y))}{f(z_n)} \right|
\]
\[
\leq \varepsilon
\]
for all \( x, y \in G \) and \( n \in \mathbb{N} \). With the use of (9) and (10), we conclude that for every \( x \in G \) there exists the following limit as \( n \) tends to infinity:
\[
(12) \quad \lim_{n \to \infty} \frac{f(z_n + x) + f(z_n - x)}{f(z_n)} =: h(x).
\]
Moreover, the so defined function \( h: G \to \mathbb{C} \) satisfies the equation
\[
f(x+y) + g(x+y) + f(-x+y) + g(-x+y) - h(x)[f(y) + g(y)] = 0, \quad x, y \in G.
\]
By interchanging \( x \) and \( y \), we obtain
\[
f(x+y) + g(x+y) + f(x-y) + g(x-y) - [f(x) + g(x)] h(y) = 0, \quad x, y \in G.
\]
Let \( F := f + g \) and \( G := \frac{1}{2} h \). Then
\[
F(x+y) + F(x-y) = 2F(x)G(y) \quad \text{for all} \quad x, y \in G.
\]
On the basis of Lemma 1, we get three possible forms of the function \( F \).

**Case 1.** Suppose \( F = 0 \). By putting \( f + g = 0 \) in (5), we obtain
\[
\left| f \left( \frac{x+y}{2} \right)^2 - f \left( \frac{x-y}{2} \right)^2 + f(x+y) + f(x-y) \right| \leq \varepsilon
\]
for all \( x, y \in G \). Now, setting \( y = 0 \), we get
\[
|f(x)| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad x \in G.
\]
This leads to a contradiction since \( f \) is unbounded.

**Case 2.** By Lemma 1 case (ii), let us assume that there exist an additive function \( a \) and a constant \( \alpha \) such that \( F = a + \alpha \). Then
\[
(13) \quad f(x) + g(x) = a(x) + \alpha \quad \text{for all} \quad x \in G.
\]
Let us take \( x = 0 \) in (13) and apply (11). We get \( \alpha = 2 \). By the inequality (5), we obtain
\[
(14) \quad \left| f \left( \frac{x + y}{2} \right)^2 - f \left( \frac{x - y}{2} \right)^2 + f(x+y) + f(x-y) - f(x)[a(y) + 2] \right| \leq \varepsilon
\]
for all \( x, y \in G \). By taking \(-y\) instead of \( y \) in (14), we infer that
\[
(15) \quad \left| f \left( \frac{x - y}{2} \right)^2 - f \left( \frac{x + y}{2} \right)^2 + f(x-y) + f(x+y) - f(x)\left[-a(y) + 2\right] \right| \leq \varepsilon.
\]
By (14) and (15) and the fact that \( a \) is odd, we get the following relation:
\[
\left| f \left( \frac{x + y}{2} \right)^2 - f \left( \frac{x - y}{2} \right)^2 + f(x+y) + f(x-y) - f(x)\left[a(y) + 2\right] \right|
\]
\[
+ \left| f \left( \frac{x - y}{2} \right)^2 - f \left( \frac{x + y}{2} \right)^2 + f(x-y) + f(x+y) - f(x)\left[-a(y) + 2\right] \right| \leq 2\varepsilon
\]
for all \( x, y \in G \). Equivalently,
\[
\left| f(x+y) + f(x-y) - 2f(x) \right| \leq \varepsilon \quad \text{for all} \quad x, y \in G.
\]
Applying Theorem 3 to the unbounded function \( f \) yields that there exist an additive function \( A \) and a constant \( \delta \) such that
\[
\left| f(x) - A(x) \right| \leq \delta \quad \text{for all} \quad x, y \in G,
\]
hence \( f = A + B \), where the function \( B \) is bounded by \( \delta \). By this fact and by (13) we get \( g = a - A - B + 2 \). Finally, we obtain (6).

Case 3. By case (iii) of Lemma 1, let us consider \( F = \beta m_o + \gamma m_e \), where the function \( m \) is exponential and \( \beta, \gamma \) are constants. Hence, from (5), we obtain
\[
(16) \quad f(x) + g(x) = \beta m_o(x) + \gamma m_e(x) \quad \text{for all} \quad x \in G.
\]
Applying (16) to \( x = 0 \) in (11), we get \( \gamma m_e(0) = 2 \). We know that if \( m \neq 0 \), then \( m(0) = 1 \) and \( m_e(0) = 1 \). In the other words, we have \( \gamma = 2 \). By (16) and (5), we get the following relation:
\[
(17) \quad \left| f \left( \frac{x + y}{2} \right)^2 - f \left( \frac{x - y}{2} \right)^2 + f(x+y) + f(x-y) - f(x)\left[\beta m_o(y) + 2m_e(y)\right] \right| \leq \varepsilon
\]
for all \( x, y \in G \). Hence, by replacing \( y \) by \(-y\) in (17), we see that
\[
(18) \quad \left| f \left( \frac{x - y}{2} \right)^2 - f \left( \frac{x + y}{2} \right)^2 + f(x+y) + f(x+y) - f(x)\left[-\beta m_o(y) + 2m_e(y)\right] \right| \leq \varepsilon
\]
for all $x, y \in G$. Summing inequalities (17) and (18) sidewise, we infer that

$$\left| f(x + y) + f(x - y) - 2f(x)m_e(y) \right| \leq \varepsilon \quad \text{for all} \quad x, y \in G.$$ 

If $m_e = 1$, then we obtain Case 2. Therefore, by Theorem 3, functions $f$ and $m_e$ satisfy the following equation:

(19) 

$$f(x + y) + f(x - y) = 2f(x)m_e(y) \quad \text{for all} \quad x \in G.$$ 

Applying (19) to (17), we see that

$$\left| f\left(\frac{x + y}{2}\right)^2 - \left(\frac{x - y}{2}\right)^2 + 2f(x)m_e(y) - f(x)[\beta m_o(y) - 2m_e(y)] \right| \leq \varepsilon,$$

for all $x, y \in G$, which is equivalent to

$$\left| f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2 + f(x)\beta m_o(y) \right| \leq \varepsilon \quad \text{for all} \quad x, y \in G.$$ 

From Lemma 2 we get what follows.

**Subcase 3.1** If the function $\beta m_o$ is bounded, then from (i) we only conclude that our function satisfies the sine functional equation. We do not get any other interesting information about the function $f$.

**Subcase 3.2** Assume that the function $\beta m_o$ is unbounded. Therefore, by (ii), there exists a function $h$ such that

(20) 

$$f(x + y) + f(x - y) = 2f(x)h(y) \quad \text{for all} \quad x \in G.$$ 

Thus, from equations (19) and (20), we obtain $h = m_e$. Furthermore, we get $f = f(0)m_e + f_o$ and the function $f_o$ satisfies the sine equation (2). It follows from (16) that

$$f(0)m_e(x) + f_o(x) + g(x) = \beta m_o(x) + 2m_e(x) \quad \text{for all} \quad x \in G.$$ 

Equivalently,

(21) 

$$g(x) = \beta m_o(x) - f_o(x) + (2 - f(0))m_e(x) \quad \text{for all} \quad x \in G.$$ 

Thus, we have proved that functions $f$ and $g$ are of the form (7).

Moreover, by Lemma 2 applied to a function $f$ for which $f(0) = 0$, we have $\beta m_o = f = f_o$. Hence, the above considerations and the equation (21) imply that $g = 2m_e$. The proof of the theorem is complete. 

**Corollary 2.** The equation (4) is not always stable.
THE STABILITY OF THE DHOMBRES-TYPE TRIGONOMETRIC FUNCTIONAL EQUATION

Proof. Assume that an unbounded function $f: G \to \mathbb{C}$ such that $f(0) = 0$ and a function $g: G \to \mathbb{C}$ satisfy the inequality (5) for all $x, y \in G$. We ask whether there exist functions $\tilde{f}, \tilde{g}: G \to \mathbb{C}$ and a constant $\delta$ such that $\tilde{f}$ and $\tilde{g}$ satisfy the equation (4) and

$$|\tilde{f}(x) - f(x)| \leq \delta \text{ and } |\tilde{g}(x) - g(x)| \leq \delta \text{ for all } x \in G.$$ 

By Theorem 4, functions $f$ and $g$ have either of the forms (6) and (8).

**Case 1.** In the case of the form (8), provided $f(0) = 0$, we define functions $\tilde{f}, \tilde{g}: G \to \mathbb{C}$ by $\tilde{f} := f$ and $\tilde{g} := g$. Therefore, for all $x, y \in G$, we get

$$\beta^2 \left[ m_o \left( \frac{x+y}{2} \right)^2 - m_o \left( \frac{x-y}{2} \right)^2 \right] + \beta \left[ m_o(x+y) + m_o(x-y) \right] = \beta m_o(x) \left[ \beta m_o(y) + 2 m_e(y) \right].$$

**Case 2.** In the case of the form (6), we have two possibilities.

**Subcase 2.1.** The function $g$ is bounded. Then the function $a - A$ is bounded and additive. Therefore, $a - A = 0$. Let us define functions $\tilde{f}, \tilde{g}: G \to \mathbb{C}$ by

$$\begin{cases} 
\tilde{f}(x) := A(x) \\
\tilde{g}(x) := 2
\end{cases} \text{ for all } x \in G.$$ 

Then the so defined functions satisfy the equation (4), i.e.,

$$A \left( \frac{x+y}{2} \right)^2 - A \left( \frac{x-y}{2} \right)^2 + A(x+y) + A(x-y) = A(x) \left[ A(y) + 2 \right]$$

for all $x, y \in G$, and $|\tilde{f} - f| = |B| \leq \delta$ and $|\tilde{g} - g| = |B| \leq \delta$ for some $\delta$.

**Subcase 2.2.** When the function $g$ is unbounded, then functions $\tilde{f}$ and $\tilde{g}$ do not exist. 

\[\square\]

**References**


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SOME PRACTICAL APPLICATIONS OF GENERATING FUNCTIONS AND LSTS

MARCEL JAKUBOWSKI

Abstract

In many various practical problems we often deal with computing distribution functions of sums of independent non-negative random variables. In applied mathematics (ex. queueing theory) we can find many formulas with Stieltjes convolutions of distribution functions of random variables of the same type. Finding convolutions on the base of definition is not easy and convenient, because there are some technical problems connected with computations. There are some interesting ways to obtain such distribution functions applying other methods. In this paper we present methods connected with applications of generating functions and Laplace-Stieltjes transforms.

1. Introduction

Assume that $\xi_1$, $\xi_2$ are two independent non-negative random variables. Distribution functions of these random variables will be denoted by $L_1(x)$ and $L_2(x)$, respectively.

For the random variable $\xi = \xi_1 + \xi_2$ we easily obtain formula for its distribution function $L(x) = P\{\xi < x\}$.

$$L(x) = P\{\xi < x\} = P\{\xi_1 + \xi_2 < x\} =$$

$$= \int_{0}^{x} P\{\xi_2 < x - u|\xi_1 \in [u, u + du]\}P\{\xi_1 \in [u, u + du]\} =$$

$$= \int_{0}^{x} P\{\xi_2 < x - u\}P\{\xi_1 \in [u, u + du]\} = \int_{0}^{x} L_2(x - u)dL_1(u).$$

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In the case of discrete independent random variables taking only integer values with distributions
\[ p_k = P\{\xi_1 = k\}, r_k = P\{\xi_2 = k\}, k = 0, 1, \ldots, \sum_k p_k = \sum_k r_k = 1 \]
we can obtain the distribution of random variable \( \xi = \xi_1 + \xi_2 \) as follows:
\[ q_k = P\{\xi = k\} = P\{\xi_1 + \xi_2 = k\} = \sum_{i=0}^k P\{\xi_1 = i, \xi_2 = k-i\} = \]
\[ = \sum_{i=0}^k P\{\xi_1 = i\} P\{\xi_2 = k-i\}. \]
Then we can finally calculate the distribution function of random variable \( \xi \) applying the formula
\[ L(x) = \sum_{k<x} q_k. \]
For the arbitrary number of independent random variables we can generalize (1), (2) by induction but final formulas may not be convenient.

For example, if we consider two independent random variables having exponential distribution with the parameter \( a \) \((a > 0)\) i.e. \( L_1(x) = L_2(x) = 1 - e^{-ax} \) for \( x > 0 \), applying integration by parts, we easily obtain
\[ L(x) = \int_0^x \left(1 - e^{-a(x-u)}\right) ae^{-au} du = \]
\[ = \int_0^x ae^{-au} du - \int_0^x ae^{-ax} du = 1 - (1 + ax) e^{-ax}. \]
So in this case we obtain 2-Erlang distribution with the parameter \( a \).

In the case of many independent random variables computation becomes very complicated because we have to use parts integration repeatedly.

Consider now two independent random variables \( \xi_1, \xi_2 \) which have geometric distribution with the parameter \( p \) \((p \in (0,1))\) i.e. \( p_k = r_k = (1-p)p^k \), \( k = 0, 1, \ldots \). Then we obtain the following result:
\[ q_k = P\{\xi_1 + \xi_2 = k\} = \sum_{i=0}^k P\{\xi_1 = i\} P\{\xi_2 = k-i\} = \]
\[ = \sum_{i=0}^k (1-p)p^i(1-p)p^{k-i} = \sum_{i=0}^k (1-p)^2p^k = (k+1)(1-p)^2 p^k. \]
In the case of \( n \) independent random variables \( (n \geq 3) \) computation also becomes more difficult because of complicated sums appearing.

Formulas with Stieltjes convolutions which can be calculated applying (1), (2) appear in applied mathematics very often. For example, they are used in theory of queueing systems with non-homogeneous customers ([3], [4], [5]).

2. LAPLACE-STIELTJES TRANSFORMATION (LST) AND ITS USEFUL PROPERTIES

Let \( \xi \) denote a non-negative random variable and \( L(x) \) its distribution function. For every complex \( q \) that has non-negative real part \( (\text{re } q \geq 0) \) we can define the following function ([1], [4]):

\[
\alpha(q) = E e^{-q\xi} = \int_{0}^{\infty} e^{-qx} dL(x). \tag{3}
\]

The function given by (3) is called Laplace-Stieltjes transformation (LST) of random variable \( \xi \).

Now we present two very interesting properties of LST that can be used in obtaining distribution functions of sums of independent random variables.

**Property 1.** Let \( \xi_1, \xi_2, \ldots, \xi_n \) denote the sequence of independent non-negative random variables and \( \alpha_1(q), \alpha_2(q), \ldots, \alpha_n(q) \) – a sequence of LST of these random variables respectively and \( \xi = \xi_1 + \xi_2 + \ldots + \xi_n \) – the sum of random variables \( \xi_1, \xi_2, \ldots, \xi_n \). Let \( \alpha(q) \) denote the LST of random variable \( \xi \). Then we obtain the following formula:

\[
\alpha(q) = \prod_{i=1}^{n} \alpha_i(q). \tag{4}
\]

**Proof.** Applying the definition of LST, in view of properties of mean value of independent random variables product, we obtain

\[
\alpha(q) = E e^{-q\xi} = E e^{-q\sum_{i=1}^{n} \xi_i} = E \prod_{i=1}^{n} e^{-q\xi_i} = \prod_{i=1}^{n} E e^{-q\xi_i} = \prod_{i=1}^{n} \alpha_i(q).
\]

\( \square \)
Property 2. Assume that $\xi$ is a non-negative random variable and denote as $L(x)$ and $\alpha(q)$ distribution function and LST of random variable $\xi$ consequently. Then we have the following formula:

$$\alpha(q) = \int_0^\infty e^{-qx} dL(x) = q \int_0^\infty e^{-qx} L(x) dx.$$  \hspace{1cm} (5)

Notice that the integral on the right side of (5) is the well known Laplace transformation of the function $L(x)$.

Proof. Calculating the integral from the left side of (5) by parts integration and the basic properties of Stieltjes integral we obtain

$$\alpha(q) = \int_0^\infty e^{-qx} dL(x) = e^{-qx} L(x)|_0^\infty - \int_0^\infty L(x) d(e^{-qx}) =$$

$$= q \int_0^\infty e^{-qx} L(x) dx.$$

Applying the two above properties we can obtain the distribution functions of sums of independent non-negative random variables. First we have to calculate LST $\alpha_i(q)$ for every random variable and in view of (4) we obtain LST

$$\alpha(q) = \prod_{i=1}^n \alpha_i(q)$$

of sum of all variables. Secondly, applying (5), we obtain Laplace transformation of the sum $l(q) = \frac{\alpha(q)}{q}$. Finally, we can use Laplace transform inversion to find distribution function of the sum. In the last step we can use residuum method or Laplace transformation tables or computer algebra systems (Mathematica environment). This method is very useful especially in the case of absolutely continuous random variables. \hfill \Box

3. Examples of calculating distribution functions of sums of independent random variables using LST

Let us consider $n$ independent random variables having exponential distribution with parameters $a_i$ ($i = 1, n$). Applying (4), (5) we obtain

$$l(q) = \frac{\prod_{i=1}^n a_i}{q \prod_{i=1}^n (a_i + q)}.$$  \hspace{1cm} (6)
Applying computer algebra systems and inverse Laplace transformation, we obtain distribution function of the sum of above random variables in the form

\[ L(x) = 1 + \sum_{i=1}^{n} \prod_{j \neq i} a_j \prod_{i \neq j} (a_i - a_j) e^{-a_i x}. \]  

(7)

In the special case of \( n \) independent random variables having exponential distribution with the same parameter \( a \) we have \( \alpha_i(q) = \frac{a}{a+q} \), \( i = 1, n \). In view of (4) and (5) we obtain formula for the Laplace transformation of the sum of these variables

\[ l(q) = \frac{a^n}{q(a+q)^n}. \]  

(8)

Applying residuum method we can obtain the distribution function in the form

\[ L(x) = 1 - e^{-ax} \sum_{i=0}^{n-1} \frac{(ax)^i}{i!}. \]  

(9)

So in this case we obtain \( n \)-Erlang distribution with the parameter \( a \).

Assume now that we have \( n \) independent random variables having uniform distribution on the interval \([a, b] \) (0 ≤ \( a \) < \( b \)) i.e. for every \( i = 1, n \) \( L_i(x) = \frac{x-a}{b-a} \) for every \( x \in [a, b] \) and \( L_i(x) = 0 \) if \( x < a \) and \( L_i(x) = 1 \) if \( x > b \). Then we have \( \alpha_i(q) = \frac{e^{-aq} - e^{-bq}}{q(b-a)} \), \( i = 1, n \). Applying (4) and (5) we obtain

\[ l(q) = \frac{(e^{-aq} - e^{-bq})^n}{q^{n+1}(b-a)^n}. \]  

(10)

Using computer algebra systems we can obtain the distribution function in the form

\[ L(x) = \left( \frac{-1}{b-a} \right)^n \sum_{l=0}^{n} \frac{(-1)^l ((b-a)l - bn + x)^n H((b-a)l - bn + x)}{l!(n-l)!}, \]  

(11)

where \( H(x) \) is the Heaviside unitstep function.

4. Generating Function (GF) and its Useful Properties

Let us consider a non-negative random variable \( \xi \) taking only integer values. Denote as \( p_k \) probability that \( \xi \) is equal to \( k \) i.e. \( p_k = P\{\xi = k\} \), \( \sum_k p_k = 1 \). Then for every complex \( z \) that satisfies condition \( |z| \leq 1 \) we can define the
following function ([2], [4]):

\[ P(z) = E z^{\xi} = \sum_{k=0}^{\infty} p_k z^k. \] (12)

Because \( P(z) \) is analytic we assume that \( P(0) = p_0 \). The function given by (12) is called the generating function (GF) of random variable \( \xi \).

Now we present two very interesting properties of GF which can be used in obtaining distribution functions of sums of independent random variables taking only integer values.

**Property 3.** Let \( \xi_1, \xi_2, \ldots, \xi_n \) denote the sequence of independent random variables taking only integer value and \( P_1(z), P_2(z), \ldots, P_n(z) \) – a sequence of GF of random variables \( \xi_1, \xi_2, \ldots, \xi_n \), respectively.

If \( \xi = \xi_1 + \xi_2 + \ldots + \xi_n \) denotes the sum of these random variables and \( P(z) \) is GF of the random variable \( \xi \), then we have the following formula:

\[ P(z) = \prod_{i=1}^{n} P_i(z). \] (13)

**Proof.** Applying the definition of GF, in view of properties of mean value of independent random variables product, we obtain

\[ P(z) = E z^{\sum_{i=1}^{n} \xi_i} = E \prod_{i=1}^{n} z^{\xi_i} = \prod_{i=1}^{n} E z^{\xi_i} = \prod_{i=1}^{n} P_i(z). \]

\[ \square \]

**Property 4.** If we have the GF \( P(z) \) of random variable \( \xi \) then we can recover distribution \( p_k = P\{\xi = k\} \) applying the formula

\[ p_k = \frac{P^{(k)}(0)}{k!}. \] (14)

**Proof.** Applying the definition of GF we obtain

\[ P^{(k)}(z) = \sum_{i=k}^{\infty} i(i-1)(i-2)\ldots(i-k+1)p_i z^{i-k} = k! \sum_{i=k}^{\infty} \binom{i}{k} p_i z^{i-k}. \] (15)

From (15) we easily obtain

\[ P^{(k)}(0) = k! p_k, \]

which confirms (14). \[ \square \]
Applying these properties we can obtain the distributions $p_k$ of sums of independent random variables which take only integer values. First we have to calculate GF $P_i(z)$ for every random variable. Then applying (13) and (14) we can obtain distribution of the sum of these variables.

5. Examples of Calculating Distributions of Sums of Independent Random Variables applying GF

Let us consider two independent random variables $\xi_1$, $\xi_2$ that are both defined by the following table:

\[
\begin{array}{c|ccc}
 x_i & 1 & 2 \\
 p_i & \frac{1}{3} & \frac{2}{3} \\
\end{array}
\]

Generating functions of both variables have the form

\[
P_1(z) = P_2(z) = \frac{1}{3}z + \frac{2}{3}z^2.
\]

Then GF of the sum $\xi_1 + \xi_2$ has the form

\[
P(z) = P_1(z)P_2(z) = \left(\frac{1}{3}z + \frac{2}{3}z^2\right)^2 = \frac{1}{9}z^2 + \frac{4}{9}z^3 + \frac{4}{9}z^4.
\]

Applying (14) we can find $p_k$ probabilities that are presented in the following table:

\[
\begin{array}{c|cccc}
 x_i & 2 & 3 & 4 \\
 p_i & \frac{1}{3} & \frac{4}{9} & \frac{4}{9} \\
\end{array}
\]

Similar computations can be leaded for the arbitrary number of independent random variables that are defined in the finite probability tables.

Assume now that we have a independent random variables $\xi_1, \ldots, \xi_n$ having Poisson distribution with parameter $\mu$ i.e. $p_k = \frac{\mu^k e^{-\mu}}{k!}$. GF of each random variable has the form: $P_i(z) = e^{-\mu(1-z)}$, $i = 1, n$. Then the GF of the sum $\xi_1 + \ldots + \xi_n$ has the form

\[
P(z) = e^{-n\mu(1-z)}.
\]

In view of (17) we obtain

\[
P^{(k)}(z) = (n\mu)^k e^{-n\mu(1-z)}.
\]
and

\[ P^{(k)}(0) = (n\mu)^k e^{-n\mu}. \]

Applying (14) we finally obtain

\[ p_k = \frac{(n\mu)^k}{k!} e^{-n\mu}. \] (18)

It follows from (18) that the sum \( \xi_1 + \ldots + \xi_n \) has the Poisson distribution with parameter \( n\mu \). If \( \xi_1, \ldots, \xi_n \) have Poisson distribution with parameter \( \mu_i \), \( i = 1, n \), making analogous computation, we can easily obtain that the sum \( \xi_1 + \ldots + \xi_n \) have Poisson distribution with the parameter \( \mu = \sum_{i=1}^n \mu_i \).

Consider now \( n \) independent random variables having geometric distribution with parameter \( p \) i.e. \( p_k = (1 - p)p^k \). Then the generating function of each variable has the form: \( P_i(z) = \frac{1}{1-pz} \), \( i = 1, n \). Then the sum of these random variables has the following GF:

\[ P(z) = (1 - p)^n (1 - pz)^{-n}. \] (19)

From (19) we easily obtain

\[ P^{(k)}(z) = n(n+1)(n+2) \ldots (n+k-1) \frac{p^k(1-p)^n}{(1-pz)^{n+k}}, \] (20)

and in view of (14) we finally obtain

\[ p_k = \binom{n + k - 1}{k} p^k (1 - p)^n. \] (21)

Analogous computations, using GF or LST, can be proceeded also for random variables which have different distribution functions.

For example, let us assume that we have two independent random variables \( \xi_1, \xi_2 \). First variable has Poisson distribution with parameter \( \mu \) and second variable has geometric distribution with parameter \( p \).

Then in view of (13) GF of the sum \( \xi_1 + \xi_2 \) has the form

\[ P(z) = \frac{e^{-\mu(1-z)}(1-p)}{1 - pz}. \] (22)

From (22), using Mathematica environment, we obtain

\[ P^{(k)}(z) = k!(1-p)e^{-\mu(1-z)} \sum_{i=0}^{k} \frac{\mu^i p^{k-i}}{i!(1 - pz)^{k+1-i}}, \] (23)
and applying (14) we finally obtain

\[ p_k = (1 - p)e^{-\mu} \sum_{i=0}^{k} \frac{\mu^i p^{k-i}}{i!} = (1 - p)p^k e^{-\mu} \sum_{i=0}^{k} \frac{(\frac{\mu}{p})^i}{i!}. \]  

(24)

6. Case of \( n \) Independent Random Vectors

Assume that \( \xi = (\xi_1, \xi_2) \) and \( \eta = (\eta_1, \eta_2) \) are two independent non-negative random vectors. Let \( L_1(x, y) \) and \( L_2(x, y) \) denote the distribution functions of these vectors, respectively. For the random vector

\[ \zeta = (\zeta_1, \zeta_2) = \xi + \eta = (\xi_1 + \eta_1, \xi_2 + \eta_2) \]

we obtain the following formula for its distribution function

\[ L(x, y) = P\{\zeta_1 < x, \zeta_2 < y\} \]

\[ L(x, y) = P\{\zeta_1 < x, \zeta_2 < y\} = P\{\xi_1 + \eta_1 < x, \xi_2 + \eta_2 < y\} = \]

\[ = \int_{0}^{x} \int_{0}^{y} P\{\xi_1 < x-u, \xi_2 < y-v|\eta_1 \in [u, u+du), \eta_2 \in [v, v+dv]\} \times \]

\[ \times P\{\eta_1 \in [u, u+du), \eta_2 \in [v, v+dv]\} = \int_{0}^{x} \int_{0}^{y} L_1(x-u, y-v) dL_2(u, v). \]  

(25)

In the case of two independent random vectors taking only integer values with distributions \( p_{ij} = P\{\xi_1 = i, \xi_2 = j\}, r_{ij} = P\{\eta_1 = i, \eta_2 = j\} \) we obtain the following formula:

\[ q_{ij} = P\{\xi_1 + \eta_1 = i, \xi_2 + \eta_2 = j\} = \]

\[ = \sum_{k=0}^{i} \sum_{l=0}^{j} P\{\xi_1 = k, \eta_1 = i-k, \xi_2 = l, \eta_2 = j-l\} = \]

\[ = \sum_{k=0}^{i} \sum_{l=0}^{j} P\{\xi_1 = k, \xi_2 = l\} P\{\eta_1 = i-k, \eta_2 = j-l\} = \]

\[ = \sum_{k=0}^{i} \sum_{l=0}^{j} p_k r_{i-k, j-l}. \]  

(26)

Then we can finally calculate the distribution function of random vector \( (\zeta_1, \zeta_2) \) applying the formula

\[ L(x, y) = \sum_{i<x} \sum_{j<y} q_{ij}. \]
Formulas (25), (26) can be generalized for the arbitrary number of non-negative independent random vectors but computations may be very complicated.

7. LST of Random Vectors and its Properties

Let \((\xi, \eta)\) denote a non-negative random vector and \(L(x, y)\) denote its distribution function. For every complex numbers \(q, s\) which satisfy the condition \(\text{Re} \, q \geq 0, \text{Re} \, s \geq 0\) we can define the LST of random vector \((\xi, \eta)\) as it follows [4]:

\[
\alpha(q, s) = E e^{-q\xi - s\eta} = \int_0^\infty \int_0^\infty e^{-qx - sy} dL(x, y).
\]

(27)

The function given by (27) is called the double LST of random vector \((\xi, \eta)\) and has the following properties.

**Property 5.** Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)\) be a sequence of independent non-negative random vectors and \(\alpha_1(q, s), \alpha_2(q, s), \ldots, \alpha_n(q, s)\) be a sequence of double LST of these random vectors respectively,

\[
(\xi, \eta) = (\xi_1, \eta_1) + (\xi_2, \eta_2) + \ldots + (\xi_n, \eta_n)
\]

— the sum of random vectors \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)\). If \(\alpha(q, s)\) is the double LST of random vector \((\xi, \eta)\), then we obtain the following formula:

\[
\alpha(q, s) = \prod_{i=1}^n \alpha_i(q, s).
\]

(28)

**Property 6.** Let us assume that \((\xi, \eta)\) is a non-negative random vector and \(L(x, y)\) and \(\alpha(q, s)\) are distribution function and double LST of random vector \((\xi, \eta)\) respectively. Then we have the following formula:

\[
\alpha(q, s) = \int_0^\infty \int_0^\infty e^{-qx - sy} dL(x, y) = qs \int_0^\infty \int_0^\infty e^{-qx - sy} L(x, y) dx dy.
\]

(29)

Let us notice that the integral on the right side of (29) is the well known double Laplace transformation of function \(L(x, y)\) ([6]).

Definition of double LST can be generalized for the arbitrary dimensions of random vectors. The properties of such generalized functions stay the same. Applying two above properties we can obtain the distribution functions of independent random vectors in a similar way as we did in the case of independent random variables.
8. Generating Function (GF) of a Random Vector and its Properties

Consider now a non-negative random vector \((\xi, \eta)\) taking only integer values and introduce the following notation:

\[ p_{ij} = P\{\xi = i, \eta = j\}, \sum_{i,j} p_{ij} = 1. \]

Then for every complex \(z_1, z_2\) that satisfy conditions \(|z_1| \leq 1, |z_2| \leq 1\) we can define the following function ([2]):

\[ P(z_1, z_2) = E^{\xi} z_1^{\eta} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij} z_1^i z_2^j. \] (30)

The function given by (30) is called the generating function (GF) of random vector \((\xi, \eta)\).

Now we present two very interesting properties of GF that can be used in obtaining distribution functions of sums of independent random vectors taking only integer values.

**Property 7.** Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)\) be a sequence of independent random vectors taking only integer values and \(P_1(z_1, z_2), P_2(z_1, z_2), \ldots, P_n(z_1, z_2)\) denote a sequence of GF of random vectors \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots, (\xi_n, \eta_n)\) respectively. If \((\xi, \eta) = (\xi_1 + \xi_2 + \ldots + \xi_n, \eta_1 + \eta_2 + \ldots + \eta_n)\) denotes the sum of these random vectors and \(P(z_1, z_2)\) – GF of the random vector \((\xi, \eta)\), then we have the following formula:

\[ P(z_1, z_2) = \prod_{i=1}^{n} P_i(z_1, z_2). \] (31)

**Property 8.** If we have the GF \(P(z_1, z_2)\) of random vector \((\xi, \eta)\) then we can recover distribution \(p_{ij} = P\{\xi = i, \eta = j\}\) applying the formula

\[ p_{ij} = \frac{1}{i! j!} \frac{\partial^{i+j} P(z_1, z_2)}{\partial z_1^i \partial z_2^j} \bigg|_{z_1 = z_2 = 0}. \] (32)

In view of those two properties we can compute the distributions of sums of independent random vectors taking only integer values. The definition of GF can be extended for the arbitrary dimensions of random vectors taking only integer values and its properties stay the same.
9. Examples of Calculating Distributions of Sums of Independent Random Vectors applying GF or LST

Assume that \((\xi_1, \eta_1), (\xi_2, \eta_2)\) are two independent random vectors taking only integer values that are defined by the following tables:

<table>
<thead>
<tr>
<th>((\xi_1, \eta_1))</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\xi_2, \eta_2))</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

The generating functions of the random vectors \((\xi_1, \eta_1)\) and \((\xi_2, \eta_2)\) have the form

\[
P_1(z_1, z_2) = \frac{5}{6} z_1^3 z_2^2 + \frac{1}{6} z_1^2 z_2^3, \quad P_2(z_1, z_2) = \frac{1}{3} z_1^3 z_2 + \frac{2}{3} z_1^4 z_2.
\]

Applying (31) we can obtain the generating function of the sum of these random vectors in the form

\[
P(z_1, z_2) = P_1(z_1, z_2) \cdot P_2(z_1, z_2) = \frac{5}{18} z_1^4 z_2^4 + \frac{11}{18} z_1^5 z_2^4 + \frac{2}{18} z_1^6 z_2^4.
\] (33)

Applying (32) and (33) we can recover the distribution of the sum. It is presented in the following table:

<table>
<thead>
<tr>
<th>((\xi_1 + \xi_2, \eta_1 + \eta_2))</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\xi_1 + \xi_2, \eta_1 + \eta_2))</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Assume now that we have \(n\) independent random vectors \((\xi, \eta)\) having the distribution defined by the formula

\[
p_{ij} = \frac{(1 - p)p^i}{e^{j!}}, \quad p \in (0, 1).
\] (34)

The generating function of the random vector \((\xi, \eta)\) has the form

\[
P(z_1, z_2) = \frac{(1 - p)e^{z_2}}{e^{(1 - p)z_1}}.
\] (35)

GF of the sum has the form

\[
Q(z_1, z_2) = \frac{e^{nz_2}(1 - p)^n}{(e - epz_1)^n}.
\] (36)

Applying (32) we obtain distribution in the form

\[
p_{ij} = \frac{1}{i!j!} \left( \frac{1 - p}{e} \right)^n p^n i^{i-1} \prod_{k=0}^{i-1} (n + k).
\] (37)
Consider finally two independent non-negative random vectors \((\xi, \eta)\) having distribution function
\[
L(x, y) = P\{\xi < x, \eta < y\} = 1 - e^{-x} - e^{-y} + e^{-(x-y)}, \quad x > 0, \quad y > 0.
\] (38)
The double LST of each vector has the form:
\[
\alpha_1(q, s) = \frac{1}{(q+1)(s+1)}.
\] (39)
Then, applying (28), the double LST of the sum of two independent vectors \((\xi, \eta)\) has the form
\[
\alpha(q, s) = \frac{1}{(q+1)^2(s+1)^2}.
\] (39)
In view of (29) we obtain formula for the double Laplace transform of this sum
\[
l(q, s) = \frac{\alpha(q, s)}{q^2s^2} = \frac{1}{q^2s^2(q+1)^2(s+1)^2}.
\] (40)
If we apply Laplace transform inversion (ex. Mathematica environment) we finally obtain formula for the distribution function of the sum
\[
L_2(x, y) = e^{-x-y} \left(2 + x + e^x(x-2)\right) \left(2 + y + e^y(y-2)\right).
\] (41)
Let us notice that the majority of calculations concern to the situation if the components of the random vectors \((\xi, \eta)\) are independent. In the discrete case i.e. the distribution of the random vector has the form \(p_{ij} = k_i l_j\), where \(k_i\) and \(l_j\) are the distributions of random variables taking only integer values. In the case of absolutely continuous random vectors the distribution function of each vector has the form \(F_{i}(x, y) = F_{1}(x)F_{2}(y)\), where \(F_{1}(x), \ F_{2}(y)\) are the distribution functions of absolutely continuous non-negative random variables.
Calculations in this case are much easier because in the discrete case the GF of the sum has the form \(P(z_1, z_2) = (P_1(z_1))^n(P_2(z_2))^n\) and it is not difficult to recover the distribution \(p_{ij}\). In the case of the sum of the absolutely continuous random vectors we have the following formula for its LST:
\[
\alpha(q, s) = (\alpha_1(q))^n(\alpha_2(s))^n
\]
and in some cases it is easier to inverse the double Laplace transform \(\frac{\alpha(q, s)}{qs}\).
If the components of the random vectors are dependent, obtaining general formulas for distribution functions of the sums is much more difficult since the distribution functions of vectors are not the products of distribution functions.
of their components and during the computations we deal with calculating more complicated sums (i.e. we have to use binomial formula), so finding general formulas is possible only in some special cases.

References


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